De este documento se sugieren los problemas:
5, 17, 21, 35, 46, 57, 102, 111, 113, 119, 128, 130, 131, 133, 152, 159, 160, 161, 162, 163, 167, 169, 172, 178, 185, 189, 196, 202, 203, 209, 212

CLASSICAL MECHANICS PROBLEM SET

Contents

1. Fundamentals of ordinary differential equations

Recommended lectures:

- Geometric interpretation of differential equations:
  - Chapter 1 from Arnold's ODE book [4].

- Euler's approximation:
  - Chapter 9 from The Feynman Lectures on Physics, volume I, [7].
  - "Euler broken lines" from Arnold's ODE book [4, Section 3.15.6].

1.1. **Geometric interpretations.** Given $V : \mathbb{R}^d \to \mathbb{R}^d$, an *autonomous* equation

$$x' = V(x)$$

establishes the way by which a particle on the position $x(t) \in \mathbb{R}^d$ at time $t \in \mathbb{R}$ moves in the *phase space* $\mathbb{R}^d$. Geometrically, the idea is that the *trajectory* of any solution are those curves with velocities matching the vector field V at every point.

Example: The system $(x', y') = (y, -x)$ has the vector field

PICTURE

and the trajectories look like

PICTURE

Notice that $(x, y) = (0, 0)$ is the only constant solution, also known as an *equilibrium* of the equation. Its trajectory is represented just by one point.

The existence theorem states that if V is continuous, then the initial value problem

$$x' = V(x) \qquad \text{subject to} \qquad x(0) = x_0.$$

has a solution. Moreover, this solution is uniquely defined in a small interval of time around 0 if V is at least Lipschitz continuous. Geometrically, this says that there one and only one curve that goes through x_0. In particular the following scenarios are impossible

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Whenever we want to consider the trajectory of all points we use the *flow* generated by V. Assuming that for every $x_0 \in \mathbb{R}^d$ there is a unique trajectory starting from x_0 *defined for every time**, the flow is the family of functions $\phi^t : \mathbb{R}^d \to \mathbb{R}^d$ (parametrized by $t \in \mathbb{R}$) such that $x(t) = \phi^t(x_0)$ solves $x' = V(x)$ subject to $x(0) = x_0$. One of the consequences of the existence and uniqueness of solutions for the initial value problem is that the flow satisfies the group identity:

$$\phi^{t+s} = \phi^t \circ \phi^s = \phi^s \circ \phi^t$$

besides $\phi^0 = id$, which is part of the given definition.

For the *non-autonomous* problem

$$x' = V(t, x)$$

it might be convenient to picture the solutions in the *extended phase space* given by $\mathbb{R} \times \mathbb{R}^d$. A standard trick is to consider the equivalent autonomous problem

$$X' = \begin{pmatrix} 1 \\ V(X) \end{pmatrix} \qquad \text{where} \qquad X = \begin{pmatrix} t \\ x \end{pmatrix}$$

If at every point we have that $W \in \operatorname{span}\{V\} \setminus \{0\}$ then $x' = W(x)$ has the same trajectories as $x' = V(x)$ (perhaps with different parametrizations). The family of lines given by $\operatorname{span}\{V\}$ is usually drawn as a small segment through the given point and it is known as the *direction field* or *slope field* if $d = 2$. This is particularly useful for visualizing the solutions of scalar problems.

Example: The logistic equation $x' = x(1-x)$ has the following slope field on the tx-plane (the extended phase space)

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*For this it suffices to satisfy the hypotheses of the existence and uniqueness theorem and control that the solutions do not escape to infinity in finite time. For instance, this happens if V is globally Lipschitz.

PICTURE
and the solutions look like
PICTURE
Notice that the equation admits two equilibria $x = 0$ and $x = 1$ which are now represented by horizontal lines in the tx-plane.

Back to the relation between $x' = V(x)$ and $x' = W(x)$ for $W \in \operatorname{span}\{V\} \setminus \{0\}$ let us mention that in some cases a convenient choice of W simplifies the problem.

Example: The Lotka-Volterra system

$$x' = x(1-y), \qquad y' = -y(1-x) \qquad \text{over } x, y > 0$$

has the same trajectories as the decoupled system

$$x' = \frac{x}{1-x}, \qquad y' = -\frac{y}{1-y}$$

Reciprocally to the embedding of a non-autonomous equation in the extended phase space, an autonomous equation can be projected to a (usually) non-autonomous equation.

Consider the system $x' = V(x)$ in $\mathbb{R}^d$. If we are interested in a solution over an interval of time for which $x_d = x_d(t)$ has an inverse $t = t(x_d)$, we can then write $x_i = x_i(t)$ (for $i \in \{1, \dots, (d-1)\}$) in terms of x_d (as an independent variable), i.e. $x_i = x_i(t(x_d))$. By the chain rule we reduce the system to one in $\mathbb{R}^{d-1}$

$$\frac{dx_i}{dx_d} = \frac{V_i}{V_d}$$

Let us emphasize the different constructions for the autonomous and the non-autonomous problem. In particular, notice that the graph of the solution in the extended phase space completely describes the solution, meanwhile the trajectory only indicates which points are visited but it does not tell the time at which they are visited.

$x' = V(x)$	$x' = V(t, x)$
Phase space: $\operatorname{Dom} V \subseteq \mathbb{R}^d$.	Extended phase space: $\operatorname{Dom} V \subseteq \mathbb{R} \times \mathbb{R}^d$.
Vector field.	Direction field.
Trajectory of a solution.	Graph of a solution.
Equilibria are points.	Equilibria are lines parallel to the t-axis.

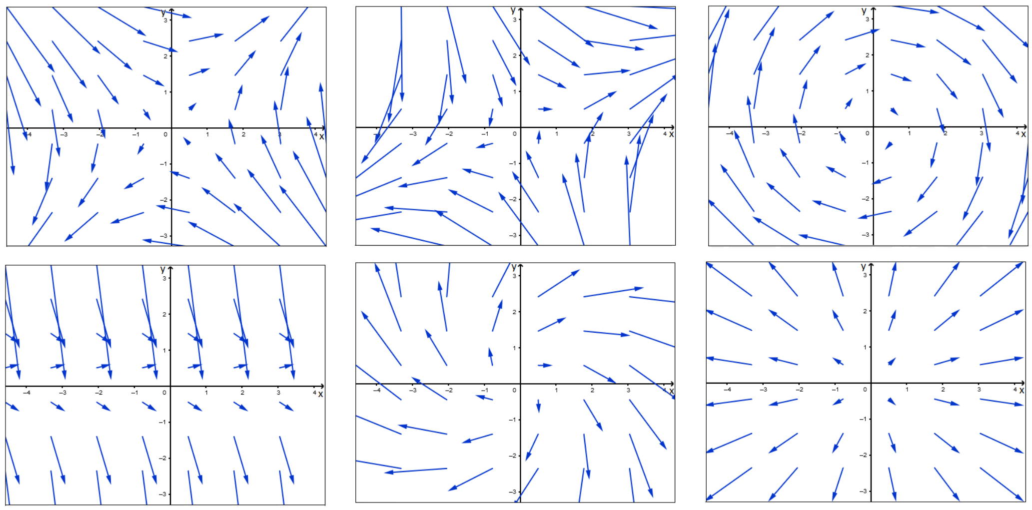
1. Match the differential equation for $(x, y) = (x(t), y(t))$ with its corresponding vector field in the xy-plane

(1) $x' = y$, $y' = x$.

(2) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

(3) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

(4) $x' = 1$, $y' = y(1-y)$.



2. Match the differential equation for $(x, y) = (x(t), y(t))$ with the graph of some of its trajectories in the xy-plane.

(1) $(x', y') = (y, x)$.

(2) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

(3) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

(4) $x' = y$, $y' = -\sin x$.

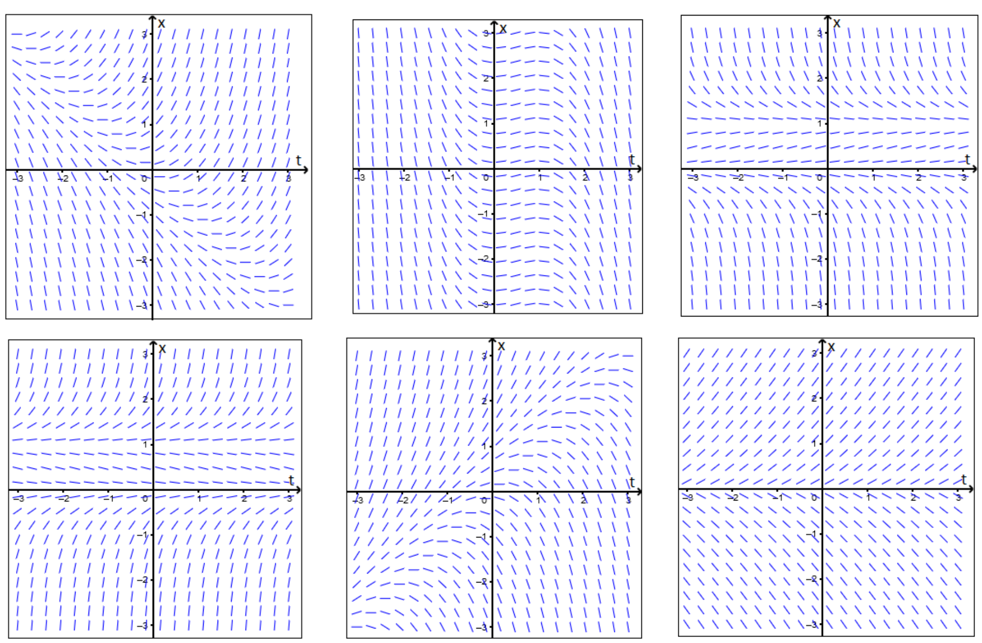
3. Match the differential equation for $x = x(t)$ with its corresponding direction field in the tx-plane

(1) $x' = x + t$.

(2) $x' = x(1 - x)$.

(3) $x' = x^{1/3}$.

(4) $x' = t(1 - t)$



4. Consider the (scalar) autonomous equation $x' = V(x)$. Explain why the slope field and the solutions remain invariant by translations parallel to the t-axis.

5. Let $x(t) = \phi(t)$ be a solution of the (implicit) autonomous equation $f(x, x') = 0$ and $y(t) = A\phi(\omega t - \delta)$ for $A, \omega, \delta \in \mathbb{R}$ $(A, \omega \neq 0)$

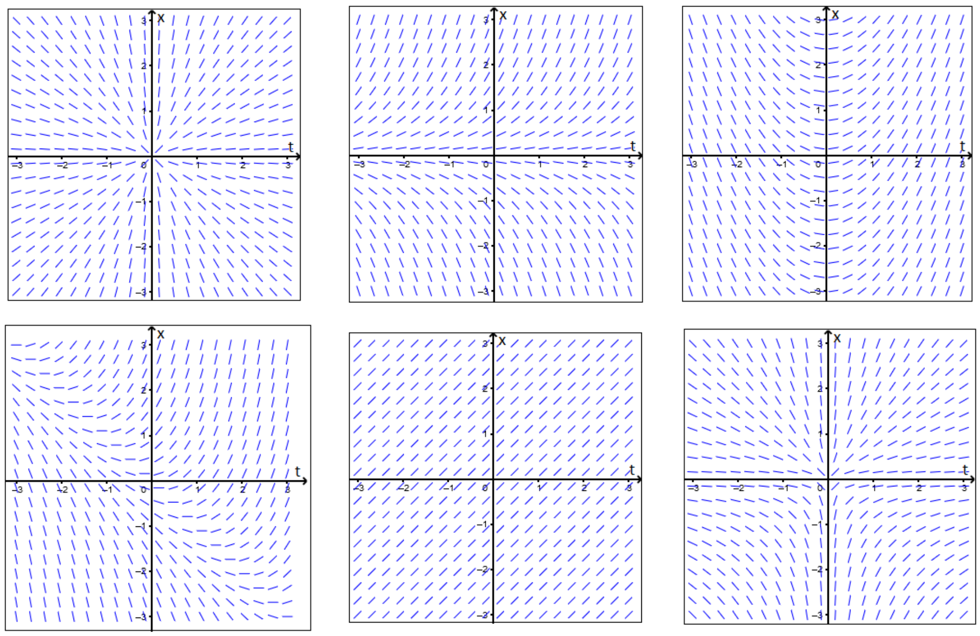
(1) Explain in geometric terms the relation between the graphs of x and y.
(2) Prove that y is a solution of $g(y, y') = 0$ where $g(y, y') = f(x/A, x'/(\omega A))$.

6. The function $x(t) = \cos t$ solves the equation $(x')^2 + x^2 = 1$. Use a change of variables to get the solution of $m(x')^2 + kx^2 = 2E$ with $x(0) = x_0$.

7. The function $x(t) = e^t/(1 + e^t)$ solves the equation $x' = x(1 - x)$. Use a change of variables to get the solution to $x' = kx(1 - x/T)$ with $x(0) = x_0 \in (0, T)$.

8. The differential equation $dy/dx = f(x, y)$ is said to be homogeneous if $f(\lambda x, \lambda y) = f(x, y)$ for all $\lambda > 0$ and $(x, y) \in \operatorname{Dom} f \subseteq \mathbb{R} \times \mathbb{R}^d$[*].

(1) Identify which one of the following slope fields correspond to homogeneous equations in the xy-plane



(2) Show that if $y = \phi(x)$ is a solution of a homogeneous equation then also $y = \lambda^{-1}\phi(\lambda x)$ is a solution for every $\lambda > 0$.

9. Are the following scenarios possible for the vector field and the trajectories of a two dimensional autonomous system?

PICTURE

10. Are the following scenarios possible for the vector field and the trajectories of a two dimensional non-autonomous system?

PICTURE

11. The vector field $V : \operatorname{Dom} V \subseteq \mathbb{R}^d \to \mathbb{R}^d$ is rotationally invariant if $V(Rx) = RV(x)$ for all $R \in SO(d)$ and $x \in \operatorname{Dom} V \subseteq \mathbb{R}^d$[†]

(1) Identify which one of the following vector fields in $\mathbb{R}^2$ are rotationally invariant

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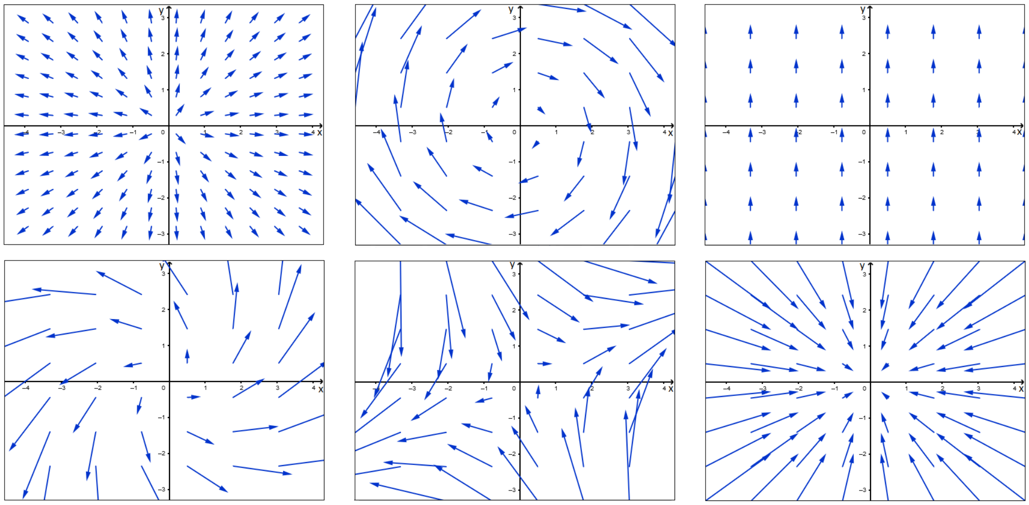
[*] $\operatorname{Dom} f$ is a cone.

[†] $\operatorname{Dom} V$ is also a rotationally invariant set. For example $\mathbb{R}^d$, a ball, or a union of rings all centered at the origin.



(2) For $d \geq 3$. Prove that the vector field $V : \operatorname{Dom} V \subseteq \mathbb{R}^d \to \mathbb{R}^d$ is rotationally invariant if and only if $V(x) = \phi(r)\theta$ where $r = |x|$ and $\theta = x/r$.

(3) For $d \geq 3$. Prove that if the vector field $V : \operatorname{Dom} V \subseteq \mathbb{R}^d \to \mathbb{R}^d$ is rotationally invariant then the differential equation $x' = V(x)$ can be reduced to solving a scalar differential equation.

12. (From [12, Chapter 1]) Consider the differential equation $x' = f(t,x)$. Suppose that

$$f(t+T,x) = f(t,x), \qquad f(t,-1) > 0, \qquad f(t,1) < 0$$

for all t. Prove that there is a periodic solution $x(t)$ for this equation taking values in the interval $(-1,1)$.

13. (Linearization) Let $\phi^t : \mathbb{R}^d \to \mathbb{R}^d$ be the flow generated by V. Show that $M(t) = D\phi^t(x_0) \in \mathbb{R}^{d\times d}$ is the solution of the linearized equation at the trajectory $\phi^t(x_0)$, i.e.

$$M' = DV(\phi^t(x_0))M, \qquad M(0) = I.$$

14. (Liouville's formula) Let $\phi^t : \mathbb{R}^d \to \mathbb{R}^d$ be the flow generated by V and $\psi^t = D\phi^t : \mathbb{R}^d \to \mathbb{R}^{d\times d}$. Let $\varphi^t = \det D\phi^t$, the factor by which ϕ^t scales the volume. Show that it satisfies

$$\varphi^{s+t} = (\varphi^s \circ \phi^t)\varphi^t \qquad \text{and} \qquad \frac{d}{dt}\varphi^t = (\operatorname{div} V \circ \phi^t)\varphi^t$$

In particular, ϕ^t preserves the volume if $\operatorname{div} V = 0$.

15. (Bendixson-Dulac Theorem) Consider a two dimensional system $x' = V(x)$ which generates a complete flow in the plane. Show that if $\operatorname{div} V$ does not vanish at any point, then the flow does not have periodic orbits (excluding fixed points).

16. Consider a two dimensional system $x' = V(x)$ which generates a complete flow in the plane. Let x_0 be part of a periodic orbit, that is to say that $\phi^T(x_0) = x_0$ for some $T > 0$. We denote $\Gamma = \{\phi^t(x_0) : t \in [0,T)\}$. Show that the periodic orbit is asymptotically stable if

$$\int_0^T \operatorname{div} V(\phi^t(x_0))dt < 0$$

In other words, there exists $\varepsilon > 0$ such that

$$|x - x_0| < \varepsilon \qquad \Rightarrow \qquad \lim_{t\to\infty} d(\phi^t(x),\Gamma) = 0 \qquad (d(x,A) = \inf_{y\in A}|x-y|)$$

1.2. **Euler's approximation 15.** To solve numerically the initial value problem $x' = Vt,(x)$, $x(t_0) = x_0$ the idea is to discretize the solution and approximate the derivative by the difference quotient. This leads to a recurrence relation to compute subsequent values which can be implemented in a computer.

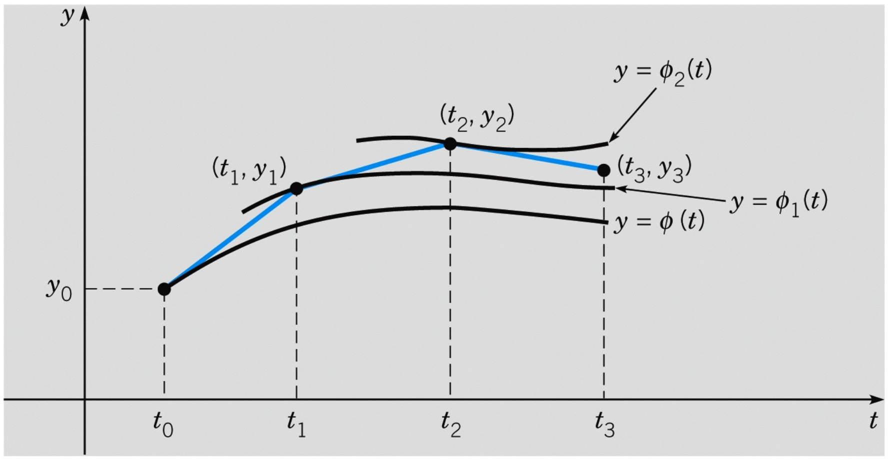


Figure 2.7.4

Example: Use Euler's approximation in the following initial value problem to guess the solution

$$x' = x \qquad x(0) = 1$$

Solution: Consider a uniform partition of $[0, T]$ with $(n+1)$ points

$$\Delta t = T/n \qquad t_i = i\Delta t$$

Then we compute a numerical approximation $x_i \sim x(t_i)$ using the following discrete version of the initial value problem*

$$\frac{x_{i+1} - x_i}{\Delta t} = x_i \qquad x_0 = 1$$

equivalently we get the recurrence

$$x_{i+1} = (1 + \Delta t)x_i \text{ with } x_0 = 1$$

It follows by an inductive argument that $x_i = (1+T/n)^i$ and guessing that $x(T) = \lim_{n\to\infty} x_n$ we compute the following ansatz for the solution

$$x(T) = \lim_{n\to\infty} (1 + T/n)^n = e^T$$

We now can easily check that $x(t) = e^t$ indeed is the solution of the desired initial value problem.

17. Read Chapter 9 from The Feynman Lectures on Physics, volume I and implement the numerical examples described there. Notice that the method for the planet model uses an scheme where the positions and velocities are taken at alternate times. Compare this numerical solution to the one where both are taken at the same time.

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*Notice that all the quantities here defined depend also on n, although this is not explicitly emphasized in order to keep the notation as simple as possible.

18. Use Euler's approximation in the following initial value problem to recover the construction of the Riemann integral

$$x' = f(t) \qquad x(t_0) = x_0 \qquad \Rightarrow \qquad x(t) = x_0 + \int_{t_0}^{t} f(s)ds$$

19. Use Euler's approximation in the following initial value problem to recover the following formula for the solution

$$x' = ax \qquad x(t_0) = x_0 \qquad \Rightarrow \qquad x(t) = x_0 e^{a(t-t_0)}$$

20. Use Euler's approximation in the following initial value problem to recover the following formula for the solution

$$x' = x + f(t) \qquad x(0) = 0 \qquad \Rightarrow \qquad x(t) = \int_0^t f(s)e^{t-s}ds$$

21. Consider the second order initial value problem

$$y'' + y = f(t), \qquad y(0) = y'(0) = 0$$

which is equivalent to the first order system

$$\begin{pmatrix} y \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \qquad \begin{pmatrix} y(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Let $T > 0$, $n \in \mathbb{N}$ (large), $\Delta t = T/n$, $f_j = f(j\Delta t)$, and $\{(y_j, v_j)^T\}$ a sequence of vectors defined by

$$\frac{1}{\Delta t}\left(\begin{pmatrix} y_{j+1} \\ v_{j+1} \end{pmatrix} - \begin{pmatrix} y_j \\ v_j \end{pmatrix} \right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_j \\ v_j \end{pmatrix} + \begin{pmatrix} 0 \\ f_j \end{pmatrix}, \qquad \begin{pmatrix} y_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(1) Prove that

$$\begin{pmatrix} y_j \\ v_j \end{pmatrix} = \sum_{k=1}^{j} \begin{pmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{pmatrix}^{k-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} f_{j-k}\Delta t.$$

(2) Check that

$$\begin{pmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{pmatrix}^{k} = \begin{pmatrix} \mathrm{Re}(z^k) & \mathrm{Im}(z^k) \\ -\mathrm{Im}(z^k) & \mathrm{Re}(z^k) \end{pmatrix} \text{ where } z = 1 + i\Delta t$$

and therefore

$$y_j = \sum_{k=1}^{j} \mathrm{Im}(1+i\Delta t)^{k-1} f_{j-k}\Delta t \qquad v_j = \sum_{k=1}^{j} \mathrm{Re}(1+i\Delta t)^{k-1} f_{j-k}\Delta t$$

(3) Justify the following limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathrm{Im}(1+i\Delta t)^{k-1} f_{n-k}\Delta t = \int_0^T \sin(T-t)f(t)dt$$

22. For sake of completeness let us review the analysis behind the previous construction of the exponential:

(1) Prove that the sequence $(1+1/n)^n$ is increasing and bounded above, therefore convergent.

(2) Let $e = \lim_{n\to\infty}(1+1/n)^n$ and $f_n(t) = (1+t/n)^n$. Show that f_n and f_n' converge locally uniformly to e^t as $n \to \infty$, therefore $(e^t)' = e^t$.

23. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ bounded and uniformly continuous. For $n \in \mathbb{N}$ consider the piecewise linear Euler's approximation defined at the vertices by

$$x_n((i+1)/n) = x_n(i/n) + (1/n)f(x_n(i/n)), \qquad x_n(0) = 0$$

Show that there is a sequence $n_k \to \infty$ such that x_{n_k} converges locally uniformly to $x \in C^1(\mathbb{R})$ that solves the initial value problem*

$$x' = f(x), \qquad x(0) = 0.$$

In particular, given that the equation satisfies the hypothesis of the uniqueness theorem (e.g. f is Lipschitz) it follows that the whole sequence x_n converges to x.

24. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ Lipschitz and bounded. For $n \in \mathbb{N}$ consider $\{x_i^n\}_{i=0}^n$ satisfying Euler's scheme

$$x_{i+1}^n = x_i^n + n^{-1}f(x_i^n)$$

with global truncation error

$$E_i^n = |x_i^n - x(in^{-1})|$$

where $x = x(t)$ is defined by $x' = f(x)$ with $x(0) = 0$. Show that for some constants $A, B > 0$ independent of $i \in \{0, 1, \ldots, (n-1)\}$ and n

$$E_{i+1}^n \leq (1 + An^{-1})E_i^n + Bn^{-2}$$

deduce that $|E_n^n| \leq Cn^{-1} + DE_0^n$ for some constants $C, D > 0$ independent of n.

1.3. **Guessing and checking solutions 21.** Sometimes we could come to the solution of a problem by intuition or just luck. We call this educated guess an *ansatz*. If the proposed solution checks the equation then the uniqueness theorem guarantees we have arrived to the desired solution.

Example: Let $x = x(t) \in \mathbb{R}^d$ be a solution of $x' = Ax$ subject to $x(0) = \xi$. If $A\xi$ is parallel to ξ, i.e. $A\xi = \lambda\xi$, we might guess from

$$x' = Ax \quad \text{ or } \quad x(t+\Delta t) \sim x(t) + \Delta t Ax(t)$$

that x remains always parallel to ξ. In other words our ansatz is that $x(t) = u(t)\xi$ for some scalar function u. Plugging this into the equation we get that u should satisfy $u' = \lambda u$ and $u(0) = 1$. Clearly this problem has the solution $u(t) = e^{\lambda t}$ and $x(t) = e^{\lambda t}\xi$ checks the initial value problem

$$x' = \lambda e^{\lambda t}\xi, \qquad Ax = e^{\lambda t}A\xi = e^{\lambda t}\lambda\xi, \qquad x(0) = \xi.$$

To check that some given functions solves a differential equation seems a bit trivial, however think about the following problem before looking at the solution.

Example: Check that the following expression gives a particular solution of the equation $x' = x + f(t)$

$$x(t) = \int_0^t f(s)e^{t-s}ds$$

Recall that you might arrive to this ansatz by Euler's approximation.

*Hint: You may to apply Arzela-Ascoli to x_n and x_n'. Even though x_n' is not continuous it can be shown that it has a modulus of continuity outside of a ball of radius $\sim 1/n$. This should be enough to apply the proof of Arzela-Ascoli in this context.

Solution 1: By the product rule and the Fundamental Theorem of Calculus

$$x' = \left(e^t \int_0^t f(s)e^{-s}ds\right)' = e^t \int_0^t f(s)e^{-s}ds + e^t f(t)e^{-t} = x + f(t)$$

Solution 2: We observe that in the expression for $x(t)$ the independent variable appears in two places, as a limit of integration and also as a parameter in the integrand. We decouple these by introducing the auxiliary function

$$\phi(u, v) = \int_0^u f(s)e^{v-s}ds \qquad \Rightarrow \qquad x(t) = \phi(u(t), v(t)) \text{ with } u(t) = v(t) = t$$

By the chain rule $x' = \partial_u \phi u' + \partial_v \phi v' = \partial_u \phi + \partial_v \phi$. The partial derivatives now are computed using the Fundamental Theorem of Calculus (for u) and interchanging the derivative with the integral (for v)

$$\partial_u \phi = \frac{d}{du} \int_0^u f(s)e^{v-s}ds = f(u)e^{v-u}, \qquad \partial_v \phi = \frac{d}{dv} \int_0^u f(s)e^{v-s}ds = \int_0^u f(s)e^{v-s}ds$$

Hence for $u(t) = v(t) = t$ we get that $\partial_u \phi = f(t)$, $\partial_v \phi = x$, and $x' = x + f(t)$.

Solution 3: Keeping in mind that $x'(t) = \lim_{\varepsilon \to 0}(x(t + \varepsilon) - x(t))/\varepsilon$

$$\begin{aligned}\frac{x(t+\varepsilon) - x(t)}{\varepsilon} &= \frac{1}{\varepsilon}\left(\int_0^{t+\varepsilon} f(s)e^{t+\varepsilon-s}ds - \int_0^t f(s)e^{t-s}ds\right) \\ &= \int_0^{t+\varepsilon} f(s)\frac{e^{t+\varepsilon-s} - e^{t-s}}{\varepsilon}ds + \frac{1}{\varepsilon}\int_t^{t+\varepsilon} f(s)e^{t-s}ds\end{aligned}$$

Then the first term converges to x and the second to $f(t)$. Note that in the second equality we "added and subtracted the same quantity" which is essentially the idea behind the proof of the product rule and also the chain rule.

Most of the following problems ask you to check some solutions or identities. We will cover how to deduce these formulas in later sections.

25. Our goal is to find a particular solution of $x'' - 2x' + x = t^2 + 1$. Given that any number of derivatives of a polynomial is still a polynomial we might guess that there exists a polynomial solution. Is this true?

26. Our goal is to find a particular solution of $x'' - 2x' + x = e^{rt}$. Given that any number of derivatives of e^{rt} is still a multiple of e^{rt} we might guess that $x(t) = Ce^{rt}$ is a solution for some appropriated C. Is this true for any r?*

27. Our goal is to find a particular solution of $mx'' + kx = F_0 \cos(\omega t)$. Given that an even number of derivatives of $\cos(\omega t)$ is still a multiple of $\cos(\omega t)$ we might guess that $x(t) = C \cos(\omega t)$ is a solution for some appropriated C. Is this true for any ω?

28. For which values of r is $x(t) = e^{rt}$ a solution of $x'' - 2x' + x = 0$?

29. For which values of ω is $x(t) = \cos(\omega t)$ a solution of $mx'' + kx = 0$?

30. For which values of γ and k is $x(t) = te^{-t}$ a solution of $x'' + \gamma x + kx = 0$?

31. Let $\gamma^2 < 4mk$. For which values of λ and μ is $x(t) = e^{\lambda t} \cos(\mu t)$ a solution of $mx'' + \gamma x' + kx = 0$?†

32. Consider the differential equation,

$$y'' - 9y = 10 \cos(t - \pi)$$

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*Hint: Be careful with zero denominators.

†Hint: $x(t) = \mathrm{Re}(e^{(\lambda + i\mu)t})$

(1) Check that

$$y(t) = Ae^{3t} + Be^{-3t} + \cos t$$

is a solution for any $A, B \in \mathbb{R}$.

(2) Determine the solution satisfying the initial conditions $y(0) = 1$, $y'(0) = 0$.

33. Consider the differential equation,

$$my'' + ky = F_0 \cos(\omega t)$$

(1) Check that if $\omega^2 \neq \omega_0^2 = m/k$ then

$$y(t) = R\cos(\omega_0 t - \delta) + \frac{F_0}{m - k\omega^2}\cos(\omega t)$$

is a solution for any $R \geq 0$ and $\delta \in [0, 2\pi)$.

(2) Check that if $\omega^2 = \omega_0^2 = m/k$ then

$$y(t) = R\cos(\omega_0 t - \delta) + \frac{F_0}{2m\omega} t \sin(\omega t)$$

is a solution for any $R \geq 0$ and $\delta \in [0, 2\pi)$.

34. Check that the following expression gives the solution of the initial value problem $x' = ax + f(t)$ with $x(t_0) = x_0$

$$x(t) = x_0 e^{a(t-t_0)} + \int_{t_0}^{t} f(s) e^{a(t-s)} ds$$

35. Check that the following expression gives a solution of the equation $x'' + x = f(t)$

$$x(t) = \int_0^t f(s)\sin(t - s) ds$$

36. Consider the differential equation,

$$y^{(4)} - y''' + y'' - y' + y = 0.$$

(1) Show that any solution also satisfies the equation $y^{(10)} = y$.

(2) Show that if for some k between 1 and 9 a solution of the differential equation also satisfies the equation $y^{(k)} = y$, then $y = 0$.

37. Check that $(x(t), y(t)) = (Ae^{3t} + Be^{t}, Ae^{3t} - Be^{t})$ solves the following system for any $A, B \in \mathbb{R}$

$$x' = 2x + y \qquad y' = x + 2y$$

38. Check that the following matrix solves the given matrix differential equation

$$x(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \qquad \Rightarrow \qquad x' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x$$

39. Let

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = E(t - t_0)\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \int_{t_0}^{t} E(t - s)\begin{pmatrix} f(s) \\ g(s) \end{pmatrix} ds \qquad E(t) = \frac{1}{2}\begin{pmatrix} e^{3t} + e^{t} & e^{3t} - e^{t} \\ e^{3t} - e^{t} & e^{3t} + e^{t} \end{pmatrix}$$

Show that this previous expression gives the solution of the initial value problem

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} \qquad \begin{pmatrix} x(t_0) \\ y(t_0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

40. Let $\Delta t > 0$, $r_\pm = 1 \pm \Delta t$, $\{f_k\}$ a sequence of real numbers, and define the sequences

$$y_n = \sum_{k=1}^{n} \frac{r_+^{k-1} - r_-^{k-1}}{2} f_{n-k} \Delta t \qquad v_n = \sum_{k=1}^{n} \frac{r_+^{k-1} - r_-^{k-1}}{2} f_{n-k} \Delta t$$

Show that these satisfy the following relations

$$\frac{y_{n+1} - y_n}{\Delta t} = v_n, \qquad \frac{v_{n+1} - v_n}{\Delta t} = y_n + f_n$$

41. Given $\Delta t \in \mathbb{R}$ and $n \in \mathbb{N}$ check that

$$\begin{pmatrix} 1 & \Delta t \\ \Delta t & 1 \end{pmatrix}^n = \frac{1}{2} \begin{pmatrix} \lambda_+^n + \lambda_-^n & \lambda_+^n - \lambda_-^n \\ \lambda_+^n - \lambda_-^n & \lambda_+^n + \lambda_-^n \end{pmatrix} \text{ where } \lambda_\pm = 1 \pm \Delta t$$

42. Let $\Delta t > 0$ and $y_n = r^n$. For which values of r does the sequence $\{y_n\}$ satisfies the recurrence relation

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta t^2} = y_n ?$$

43. Let $\Delta t > 0$ and $y_n = (a + ib)^n \in \mathbb{C}$. For which values of a and b does the sequence $\{y_n\}$ satisfies the recurrence relation

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{\Delta t^2} = -y_n ?$$

44. Let

$$\frac{dy}{dx} = f(x, y) = -\frac{\partial_x I(x, y)}{\partial_y I(x, y)}$$

show that $I(x, y)$ remains constant along the trajectory*.

45. Consider an α-homogeneous potential U ($U(\lambda x) = \lambda^\alpha U(x)$ for any $\lambda > 0$). Given the scaling symmetry of the potential we could expect a corresponding symmetry in the equations of motion. Check that if $x = x(t)$ is a solution of the system $x'' = -DU$ then also $\tilde{x} = \beta x(\gamma t)$ is a solution for any $\beta > 0$ and some γ depending on α and β.

46. Consider Newton's equation $mx'' = f(x)$ in $\mathbb{R}^3$. Show that if $f(x) = \phi(r)\theta$ is a central force then the angular momentum $M = x \times mx'$ is conserved.

47. Consider the n-body problem in $\mathbb{R}^d$

$$m_i x_i'' = \sum_{\substack{j=1,\dots,n \\ j \neq i}} -U'(r_{ij})\theta_{ij}, \qquad r_{ij} = r_{ji} = |x_i - x_j|, \qquad \theta_{ij} = -\theta_{ji} = (x_i - x_j)/r_{ij}$$

under the assumption of no collisions ($r_{ij} > 0$).

(1) Show that the energy

$$E = \frac{1}{2} \sum_{i=1,\dots,n} m_i (x_i')^2 + \frac{1}{2} \sum_{\substack{i,j=1,\dots,n \\ i \neq j}} U(r_{ij}) \text{ is conserved.}$$

(2) Show that the linear momentum

$$P = \sum_{i=1,\dots,n} m_i x_i' \text{ is conserved.}$$

*Hint: A differentiable function is constant if its derivative is...

(3) For $d = 3$, show that the angular momentum

$$M = \sum_{i=1,\ldots,n} x_i \times m_i x_i' \text{ is conserved}$$

48. Consider Euler's system of equations

$$I_1\Omega_1' = (I_2 - I_3)\Omega_2\Omega_3, \quad I_2\Omega_2' = (I_3 - I_1)\Omega_3\Omega_1, \quad I_3\Omega_3' = (I_1 - I_2)\Omega_1\Omega_2$$

Show that

$$I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2 = 2E \text{ and } I_1^2\Omega_1^2 + I_2^2\Omega_2^2 + I_3^2\Omega_3^2 = M$$

are conserved quantities.

49. Let $x = x(t) \in \mathbb{R}^3$ such that x'' is always perperdicular to x and x'. Show that $|x(t)|^2 = |x(t_0) + x'(t_0)t|^2$.

50. (Putnam 1973 - A5) Consider the initial value problem

$$x' = yz, y' = zx, z' = xy \text{ with } (x(0), y(0), z(0)) = (x_0, y_0, z_0)$$

(1) If two of these coordinates are zero, show that the object is stationary for all t.
(2) If $(x_0, y_0, z_0) = (1, 1, 0)$, show that at time t, $(x, y, z) = (\sec t, \sec t, \tan t)$.
(3) If $(x_0, y_0, z_0) = (1, 1, -1)$, show that at time t, $(x, y, z) = (1/(1 + t), 1/(1 + t), -1/(1 + t))$.

51. (From [9, Section 3.4]) Show that both functions

$$y_1(x) = \int_0^\infty \frac{e^{-tx}}{1+t^2}dt \text{ and } y_2(x) = \int_0^\infty \frac{\sin t}{t+x}dt$$

satisfy the differential equation $y'' + y = 1/x$. Prove that these functions are equal.

52. Our goal is to find a particular solution of $r^2u'' - 2ru' + u = r^\alpha$ for $r > 0$. Given that $r(r^\alpha)'$ and $r^2(r^\alpha)''$ are multiples of r^α we might guess that $u(r) = Cr^\alpha$ is a solution for some appropriated C. Is this true for any α?

53. For which values of α is $u(r) = r^\alpha$ a solution of $r^2u'' - 2ru' + u = 0$?

54. For which values of α and β is $u(r) = r^\alpha \cos(\beta \ln r)$ a solution of $r^2u'' - ru' + 2u = 0$?*

55. Check that the following expression gives the solution of the initial value problem $ru' = \alpha u + f(r)$ over $r > 0$ with $u(1) = u_1$

$$u(r) = u_1 r^\alpha + \int_1^r f(s)\left(\frac{r}{s}\right)^\alpha \frac{ds}{s}$$

56. Check that the following expression gives the solution of the initial value problem $x' = a(t)x + f(t)$ with $x(t_0) = x_0$

$$x(t) = x_0 e^{\int_{t_0}^t a(s)ds} + \int_{t_0}^t f(s)e^{\int_s^t a(r)dr}ds$$

57. (Chebyshev polynomials) Let $T_0(x) = 1$, $T_1(x) = x$, and for $n \geq 1$, $T_{n+1}(x)$ is determined by the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

(1) Check that $T_n(\cos\theta) = \cos(n\theta)$.

*Hint: $u(r) = \mathrm{Re}(e^{(\alpha+\beta i)\ln r})$

(2) Check that

$$T_n(x) = \operatorname{Re}\left(x + i\sqrt{1-x^2}\right)^n = \frac{(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n}{2}$$

(3) Check that $y(x) = T_n(x)$ solves

$$(1-x^2)y'' - xy' + n^2 y = 0$$

2. Methods to integrate ordinary differential equations

Recommended lectures:

- Chapter 4 from The Feynman Lectures on Physics, volume I, [7].
- Chapter 3 from Tenenbaum and Pollard's book [14].
- Section 2.12 from Arnold's ODE book [4].
- Chapter 2 from Arnold's mechanics book [2].

In the first sections of this part we focus on first order and scalar equations. Later on we move on to some first order systems. Keep in mind that higher order equations can always be presented as an equivalent first order system.

2.1. Scalar autonomous equations 54.

For the scalar equation $x' = f(x)$ we have that, if $x(t_0) = x_0$ is not an equilibrium of the equation, then $f(x_0)$ is either positive or negative and then the velocity x' will always have the same sign, as a consequence of the Uniqueness Theorem. This means that $x = x(t)$ is strictly monotonone, which for functions of one variable implies that it has an inverse $t = t(x)$. By the chain rule

$$\frac{dt}{dx} = \frac{1}{x'(t(x))} = \frac{1}{f(x)} \quad \text{with} \quad t(x_0) = t_0 \qquad \Rightarrow \qquad t(x) - t_0 = \int_{x_0}^{x} \frac{d\xi}{f(\xi)}$$

Example: Find the velocity of a mass moving vertically upon which acts the force of gravity and air drag

$$mv' = -mg - \gamma v, \qquad v(t_0) = v_0$$

Solution: The equilibrium of this equations happens when $v_0 = -mg/\gamma$. Otherwise we can assume that either $v(t) > -mg/\gamma$ or $< -mg/\gamma$. Let us do the case $v(t) > -mg/\gamma$ and the other one is just similar.

$$t(v) - t_0 = \int_{v_0}^{v} \frac{d\xi}{-g - (\gamma/m)\xi} \underset{u = g + (\gamma/m)\xi}{=} -\frac{\gamma}{m}\int_{g+(\gamma/m)v_0}^{g+(\gamma/m)v} \frac{du}{u} = -\frac{\gamma}{m}\ln\left(\frac{g+(\gamma/m)v}{g+(\gamma/m)v_0}\right)$$

Solving for v in terms of t and the other constants

$$v(t) = \left(v_0 + \frac{mg}{\gamma}\right)e^{-(\gamma/m)(t-t_0)} - \frac{mg}{\gamma}$$

Notice that this formula cover all the cases for v_0. Also $v(t) \to -mg/\gamma$ as $t \to \infty$, meaning that the velocity converges to the equilibrium independently of the initial condition, a property easily observed in physical experiments.

Example: The law of conservation of energy states that

$$\frac{m}{2}(x')^2 + U(x) = E$$

If we restrict the problem to $\{x' > 0\}$ we get the autonomous equation

$$x' = \sqrt{(2/m)(E - U(x))} \qquad \Rightarrow \qquad t - t_0 = \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{d\xi}{\sqrt{E - U(x)}}$$

58. Solve the following initial value problem modeling a fall with nonlinear damping

$$mv' = -mg - \gamma v^2 \qquad v(t_0) = v_0$$

59. Solve the following initial value problem for the law of conservation of energy in Hooke's spring law for all t for which $x'(t) > 0$

$$\frac{m}{2}(x')^2 + \frac{k}{2}x^2 = E \qquad x(0) = 0 \qquad x'(0) > 0$$

Check that the actual solution found can be extended for all t regardless of the assumption on the sign of the velocity.

60. (Landau-Lifschitz, Sec. 12) Consider an unknown even potential $U(x) = U(-x)$ such that U is increasing in $[0, \infty)$ and $U(0) = 0$. Show that U can be recovered if the period of motion $T = T(E)$ is a known function of the total energy by inverting the following relation for $U \geq 0$

$$x(U) = \frac{1}{2\pi\sqrt{2m}} \int_0^U \frac{T(E)dE}{\sqrt{U - E}}$$

61. Solve the following initial value problem modelling the logistic growth of a population

$$x' = x(1 - x) \qquad x(t_0) = x_0.$$

Warning: Identify first the equilibria and then consider different cases according to the position of x_0 with respect to these equilibria. It might be helpful to sketch the slope field to get an idea of how the solutions behave.

62. (Berkeley Qualifying - Spring 1981) Consider the system

$$x' = y + x(1 - x^2 - y^2), \qquad y' = -x + y(1 - x^2 - y^2)$$

Show that if $(x(0), y(0)) \neq (0, 0)$ then the trajectory approaches the unit circle.

63. Prove that a solution the autonomous equation $x' = f(x)$ is increasing whenever it takes values over the set $\{f > 0\}$.

64. Prove that a solution the autonomous equation $x' = f(x)$ is convex whenever it takes values over the set $\{ff' > 0\}$. Moreover, inflection points only appear over $\{f' = 0, f \neq 0\}$.

65. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f(x) = 1 + \int_0^x f(t)dt$$

66. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y) = \frac{f(x) + f(y)}{1 + f(x)f(y)}$$

67. (The presentation we made at the beginning of this section on how to integrate an autonomous equation is only formal. This means that we need to justify under which conditions on f is the integration and the inversions valid.) Show that if f is continuous over an interval $I = (a, b)$ containing x_0 then the initial value problem $x' = f(x)$, $x(t_0) = x_0$ has a solution in some time interval of the form $(t_0 - \varepsilon, t_0 + \varepsilon)$.

68. Show that if f is continuous over an interval $I = (a, b)$ containing x_0 and $f(x_0) \neq 0$ then the initial value problem $x' = f(x)$, $x(t_0) = x_0$ has a **unique** solution in some time interval of the form $(t_0 - \varepsilon, t_0 + \varepsilon)$.

69. Check that the initial value problem $x' = x^{1/3}$, $x(0) = 0$ has at least two solutions.

70. Consider $x' = f(x)$ with $f(x) \leq C|x|$. Show that $x = 0$ is the unique solution of $x' = f(x)$ with $x(0) = 0$. In general deduce that if f is Lipschitz continuous then the initial value problem $x' = f(x)$ with $x(t_0) = x_0$ has a unique solution in some time interval of the form $(t_0 - \varepsilon, t_0 + \varepsilon)$.

71. Consider $x_1 = x_1(t)$ and $x_2 = x_2(t)$ two solutions of $x' = f(x)$. Show that if f is Lipschitz then

$$|x_1(t) - x_2(t)|^2 \leq |x_1(0) - x_2(0)|^2 e^{C|t|}$$

In particular this implies the uniqueness of solutions for the corresponding initial value problem.

2.2. **Separable equations 68.** A mnemotechnic trick to integrate an autonomous equation is to treat dx/dt as a fraction and see that we can separate the variables in the equation. All the x's on one side (including dx), and all the t's in the other (including dt)

$$\frac{dx}{dt} = f(x) \qquad \Leftrightarrow \qquad \frac{dx}{f(x)} = dt$$

Then, integrating both sides one gets $t = \int dx/f(x)$ and then proceed to invert the relation.

Clearly this method of separating the variables admits more general settings than the autonomous equation.

Example: (Berkeley Qualifying - Fall 87) Find a curve C in $\mathbb{R}^2$, passing through the point $(3, 2)$, with the following property: Let $L(x_0, y_0)$ be the segment of the tangent line to C at (x_0, y_0) which lies in the first quadrant. Then each point (x_0, y_0) of C is the midpoint of $L(x_0, y_0)$.

Solution: Let us assume that the desired curve is given by $y = f(x)$. The equation of the line containing $L(x_0, y_0)$ is $y - y_0 = f'(x_0)(x - x_0)$ which intersect the y-axis at $y = y_0 - f'(x_0)x_0 = f(x_0) - f'(x_0)x_0$. To say that (x_0, y_0) bisects $L(x_0, y_0)$ is equivalent to say that the previous y-intersect is at twice the height of y_0 which gives us the desired differential equation for f

$$f - xf' = 2f \qquad \Rightarrow \qquad \frac{df}{f} = -\frac{dx}{x} \qquad \Rightarrow \qquad \ln f = -\ln x + C \qquad \Rightarrow \qquad f(x) = Cx^{-1}$$

Given that the point $(3, 2)$ is on the curve we get that $C = 6$ and the desired curve is given by $y = 6x^{-1}$.

72. (Tenenbaum - Chp. 3) Find the family of curves which has the property that the segment of a tangent line drawn between a point of tangency and the y-axis is bisected by the x axis.

73. Find all the curves perpendicular to $x^2 - y^2 = C$ with $C \in \mathbb{R}$.

74. The differential equation $dy/dx = f(x, y)$ is said to be homogeneous if $f(\lambda x, \lambda y) = f(x, y)$ for all $\lambda > 0$ and $(x, y) \in \mathbb{R}^2$. Show that an homogeneous equation becomes exact after the change of variables $u = y/x$ in the domain $x > 0$.

75. The differential equation $dy/dx = f(x, y)$ is said to be quasi-homogeneous of degree (α, β) if $f(\lambda^\alpha x, \lambda^\beta y) = \lambda^{\beta-\alpha} f(x, y)$ for all $\lambda > 0$ and $(x, y) \in \mathbb{R}^2$. Show that a quasi-homogeneous equation becomes exact after the change of variables $u = y^\alpha / x^\beta$ in the domain $x > 0$.

76. Give a formal justification of the method of separation of variables using the chain rule*.

2.3. **Exact equations 73.** One thing we learn from solving autonomous and separable equations is that given the differential equations we can in some cases find a function depending on the dependent and independent variable which remains constant along the trajectory. These functions are known as first integrals.

We proceed now in the opposite direction. Start with the first integral as a conserved quantity $I(t, x(t)) = C$ and then differentiate. We get in this way an equation for which we know that for sure there is a first integral. Namely

$$M(t, x) + N(t, x)x' = 0 \qquad M = \partial_t I, \qquad N = \partial_x I$$

(This is what your professors might do to give you some homework and exam problems.)

We say that the equation $M + Nx' = 0$ is exact if there exist some I such that $M = \partial_t I$, and $N = \partial_x I$. Notice that not every pair M, N will do the job. Indeed, from Clairaut's Theorem (commutativity of the partial derivatives) we know that $\partial_x M = \partial_t N$ is a condition necessary for the equation to be exact. Poincaré's Lemma guarantees that this condition is also sufficient if the domain of the differential equation does not have holes in it (i.e. it is simply connected).

Example: Find all the curves perpendicular to the family of curves $x^4 - 6x^2y^2 + y^4 = C$ with $C \in \mathbb{R}$.

Solution: The idea is to get a differential equation for the desired curves. We know that the gradient of $f(x, y) = x^4 - 6x^2y^2 + y^4$ is a vector perpendicular to the level sets of f at every point. Hence we can use this gradient to compute the slope of the desired curves

$$Df = (4x^3 - 12xy^2, -12x^2y + 4y^3)$$

Then we just have to solve

$$\frac{dy}{dx} = \frac{-12x^2y + 4y^3}{4x^3 - 12xy^2} \qquad \Leftrightarrow \qquad (-12x^2y + 4y^3)dx + (-4x^3 + 12xy^2)dy = 0$$

One easily checks that the equation is exact then we can find I by integrating $\partial_x I = -12x^2y + 4y^3$ with respect to x. For the purpose of integrating with respect to x we treat y as a constant

$$I(x, y) = \int (-12x^2y + 4y^3)dx = -4x^3y + 4y^3x + h(y)$$

Notice that the constant of integration has to be replaced by a function of y because, as we just say in the previous sentence, for the purpose of integrating with respect to x, y is a constant.

To figure out the function h we now use

$$-4x^3 + 12xy^2 = \partial_y I = -4x^3 + 12xy^2 + h'(y) \qquad \Leftarrow \qquad h'(y) = 0$$

*Hint: We already did something similar for the autonomous equation.

which holds if h is constant. Any constant will do, this is just an auxiliary function. My favorite one today is $h = 0$. Then $I(x, y) = -4x^3y + 4y^3x$ and

$$-4x^3y + 4y^3x = C$$

gives the desired family of curves.

Warning: The first integral $I(x, y) = C$ says that the trajectory of the solution is contained in the level set of I but it does not necessarily visit all points in it. Many examples can be given by just letting $\{I = C\}$ be a disconnected set.

77. Find all the curves perpendicular to $x^3 - 3xy^2 = C$ with $C \in \mathbb{R}$.

78. Find all the curves perpendicular to $\mathrm{Re}(x + iy)^n = C$ with $C \in \mathbb{R}$.

79. The problem of finding all the curves perpendicular to $u(x, y) = C$ gives an exact equation if u is a harmonic function*.

80. The exact equation $2vdv + \sin x dx = 0$ admits the first integral $I(x, v) = v^2 - \cos x$.
   (1) Prove that for $C \geq 1$ a trajectory in $\{I = C\}$ visits all points in this set.
   (2) Prove that for $C \in [-1, 1]$ a trajectory in $\{I = C\}$ does not visit all points in this set.

2.4. **Integrating factor and linear equations 77.** The main idea is that the possibly non exact equation $M + Nx' = 0$ might become exact after multiplying it by an appropriated auxiliary function. The main example where this technique applies are the linear equations

$$x' = a(t)x + f(t)$$

Let us illustrate the technique in the case of constant coefficients.

Example: The following equation is not exact for $a \neq 0$

$$x' = ax + f(t) \qquad \Leftrightarrow \qquad M + Nx' = 0 \text{ with } M = ax + f(t), N = -1$$

However, it might become exact if we multiply it by an appropriated auxiliary function $\mu = \mu(t)$. So

$$\mu(ax + f(t)) + (-\mu)x' = 0 \text{ is exact if } \partial_x(\mu(ax + f(t)) = \partial_t(-\mu) \Leftrightarrow \mu' = -a\mu$$

This auxiliary equation for μ is separable and we find out that $\mu(t) = e^{-at}$ is a solution (any solution will do the job). We get in this way an equivalent exact equation which can be integrated. Notice that by the product rule $\mu x' - \mu a x = \mu x' - \mu' x = (\mu x)'$ so that

$$e^{-at}x(t) - e^{-at_0}x_0 = \int_{t_0}^t f(s)e^{-as}ds \qquad \Leftrightarrow \qquad x(t) = x_0e^{a(t-t_0)} + \int_{t_0}^t f(s)e^{a(t-s)}ds$$

81. Use an integrating factor to deduce the following formula for the solution for $r > 0$

$$ru' = \alpha u + f(r) \qquad u(1) = u_1 \qquad \Rightarrow \qquad u(r) = u_1r^\alpha + \int_1^r f(s)\left(\frac{r}{s}\right)^\alpha \frac{ds}{s}$$

82. Use an integrating factor to deduce the following formula for the solution

$$x' = a(t)x + f(t) \qquad x(t_0) = x_0 \qquad \Rightarrow \qquad x(t) = x_0e^{\int_{t_0}^t a(s)ds} + \int_{t_0}^t f(s)e^{\int_s^t a(r)dr}ds$$

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*$\Delta u = \partial_x^2 u + \partial_y^2 u = 0$

83. (Putnam 1988 - A2) A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

84. The Bernoulli differential equation $y' + p(t)y = q(t)y^\alpha$ becomes linear after the change of variables $u = y^{1-\alpha}$. Use this hint to deduce the formula for the general solution.

2.5. **Trajectories for two dimensional autonomous systems 81.** Here we consider systems of the form

$$\frac{dx}{dt} = f(x, y) \qquad \frac{dy}{dt} = g(x, y)$$

Geometrically we can plot the vector field $(x, y) \mapsto (f, g)$ to understand the behavior of the solutions which should be given by curves with velocities matching the vector field.

If $x = x(t)$ has inverse $t = t(x)$ then we can write y as a function of x, namely $y(x) = y(t(x))$. By the chain rule

$$\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}$$

which is an scalar equation that determines the trajectory of the curve.

2.5.1. *Linear homogeneous systems with constant coefficients.* These are of the form

$$\frac{dx}{dt} = ax + by \qquad \frac{dy}{dt} = cx + dy$$

The equation for the trajectory can be written as

$$\frac{dy}{dx} = \frac{cx + dy}{ax + by} = f(x, y)$$

Notice that this may result in a separable or exact equation under certain assumptions on the coefficients. In general we notice that the equation is homogeneous in the sense that the right-hand side is invariant by dilations, i.e. $f(\lambda x, \lambda y) = f(x, y)$ for any $\lambda > 0$. The change of variables $u = y/x$ gives the equivalent separable equation

$$u + x\frac{du}{dx} = \frac{dy}{dx} = f(x, y) = f(1, u) \qquad \Leftrightarrow \qquad \frac{du}{f(1, u) - u} = \frac{dx}{x}$$

Example: Compute the trajectories of the system $x' = x + y$, $y' = x - y$.
Solution 1: $y = y(x)$ satisfies

$$\frac{dy}{dx} = \frac{x - y}{x + y}$$

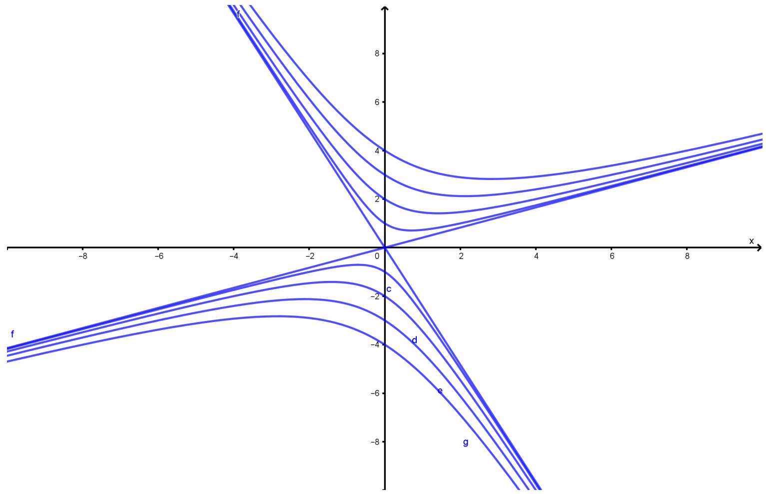
Taking $u = y/x$

$$u + x\frac{du}{dx} = \frac{1 - u}{1 + u} \qquad \Leftrightarrow \qquad \frac{1 + u}{1 - 2u - u^2}du = \frac{1}{x}dx$$

Then we compute the integral using a partial fraction decomposition and after some algebraic manipulations find that

$$x\sqrt{u^2 + 2u - 1} = C \qquad \Leftrightarrow \qquad y^2 + 2xy - x^2 = C$$

which is the equation of a rotated hyperbola.



Solution 2: Luckily the equation $(x-y)dx-(x+y)dy=0$ is exact and the method used to integrate exact equations also gives the relation $y^2+2xy-x^2=C$.

85. Sketch the vector field and compute the trajectories of the system $x'=x$, $y'=y$.

86. Sketch the vector field and compute the trajectories of the system $x'=y$, $y'=x$.

87. Sketch the vector field and compute the trajectories of the system $x'=y$, $y'=-x$.

88. Sketch the vector field and compute the trajectories of the system

$$x'=2x+y, \qquad y'=x+2y$$

89. Sketch the vector field and compute the trajectories of the system

$$x'=x-y, \qquad y'=x+y$$

90. Consider

$$\frac{dx}{dt}=ax+by \qquad \frac{dy}{dt}=cx+dy \qquad \Rightarrow \qquad \frac{dy}{dx}=\frac{cx+dy}{ax+by}$$

(1) Under which conditions on the coefficients is the equation for dy/dx separable?
(2) Under which conditions on the coefficients is the equation for dy/dx exact?

91. Solve the following initial value problem and sketch the trajectory in the xy-plane

$$x'=yx+y, \qquad y'=x^2+x, \qquad x(0)=1, \qquad y(0)=0.$$

92. Sketch the vector field and compute the trajectories of the Lotka-Volterra system

$$x'=kx-axy \qquad y'=-ly+bxy \qquad a,b,k,l>0$$

Show that each one of these trajectories in the first quadrant bound a convex set containing the equilibrium $(x,y)=(l/b,k/a)$.

93. (Putnam 1952 - B2) Find the surface comprising the curves which satisfy

$$\frac{x'}{yz}=\frac{y'}{zx}=\frac{z'}{xy}$$

and which meet the circle $x=0$, $y^2+z^2=1$.

94. (Putnam 1953 - B3) Solve the initial value problem*

$$y'=z(y+z)^k, \qquad z'=y(y+z)^k, \qquad y(0)=1, \qquad z(0)=0$$

*Hint: Consider also a differential equation for $y+z$

2.6. **Conservative forces 91.** Newton's equation $mx'' = f(x)$ can be written as the system

$$x' = v \qquad mv' = f(x) \qquad \Rightarrow \qquad \frac{dv}{dx} = \frac{f(x)}{mv}$$

The equation for $v = v(x)$ is separable and admits a first integral

$$mvdv - f(x)dx = 0 \qquad \Rightarrow \qquad \frac{m}{2}v^2 + U(x) = E$$

where U is known as the potential energy and it is determined up to constants by

$$U'(x) = -f(x)$$

You might have recognized that the conserved quantity above is the law of conservation of energy.

The energy level curves $\frac{m}{2}v^2 + U(x) = E$ in the xv-plane can be sketched from the graph of U. The idea is to keep in mind the trade off between the kinetic and potential energy. Let us illustrate this with an example and refer to Arnold's page 17-20 for a detailed explanation.

PICTURES(double well)

In higher dimensions a force field is conservative if $f = -DU$. For example, the spherical harmonic oscillator $f(x) = -kx$ has the potential $U(x) = (k/2)|x|^2$.

Another example of conservative fields are the central forces. These are vector fields which are invariant by rotations. That is to say that

$$f(x) = \phi(r)\theta \text{ where } r = |x|, \theta = x/r$$

By the symmetry of the force we can guess that the potential is a function that depends only on r, i.e. $U(x) = u(r)$. Then $DU = u'(r)\theta$ and we just have to take u as a primitive of $-\phi$ to get the desired potential.

On the other hand not every $f \in C^1(\Omega)$ is conservative ($\Omega \subseteq \mathbb{R}^d$). By the commutativity of the partial derivatives we have that if $f = -DU$ then $\partial_i f_j = \partial_j f_i$. Poincaré's Lemma tells us that this condition is sufficient if Ω is simply connected*.

The main property of conservative forces is that they also have a law of conservation of energy. The first step of the proof in the one dimensional argument now does not work because we can not divide vectors. However the next step which is multiplying the denominators has the analogue of taking the dot product with the velocity

$$mx'' \cdot x' = f(x) \cdot x'$$

Now we notice that both sides above are exact derivatives. The left-hand side is the derivative of the kinetic energy $(m/2)|x'|^2$, and the right hand side is the derivative of $-U(x)$. Putting the terms together

$$\left(\frac{m}{2}|x'|^2 + U(x)\right)' = 0 \qquad \Rightarrow \qquad \frac{m}{2}|x'|^2 + U(x) = E$$

Finally let us mention how to compute the potential U from the force f. To do this one fixes a point $x_0 \in \Omega$ and for each x consider a curve $\gamma \in C^1([0,1] \to \Omega)$ that connects x_0 with x, i.e. $\gamma(0) = x_0$ and $\gamma(1) = x$. Then we compute U by the following line integral

$$U(x) = -\int_0^1 f(\gamma(t)) \cdot \gamma'(t)dt$$

*Ω is simply connected if it is connected and every closed curve can be continuously contracted to a point without leaving Ω. e.g. $\mathbb{R}^d \setminus \{0\}$ is simply connected for $d \geq 3$ but not for $d \in \{1,2\}$.

Poincaré's Lemma guarantees that this construction does not depend on the curve γ. On the other hand, changing the base point x_0 only modifies U by an additive constant.

Example: Let $\Omega = \mathbb{R}^d$ and $f(x) = -kx$. Clearly Ω is simply connected and $\partial_i f_j = \partial_j f_i$. Let $x_0 = 0$ and $\gamma(t) = tx$, then

$$U(x) = -\int_0^1 -k(tx) \cdot x dt = \frac{k}{2}|x|^2$$

Just to double check, verify that $DU(x) = kx = -f(x)$.

95. Match the graph of the potential with its corresponding energy level curves in the xv-plane assuming $m = 2$

96. Match the potential with its corresponding energy level curves in the xv-plane assuming $m = 2$

(1) $U(x) = x(x^2 - 1)$ (3) $U(x) = (x^2 - 1)^2$
(2) $U(x) = -\cos x$ (4) $U(x) = -2\cos x$

97. Show that all trajectories of the system $mx'' = f(x) = -DU(x)$ with total energy E lie in the region $\{U \leq E\}$.

98. Assume that $mx'' = -DU(x) - \gamma x'$, with $\gamma > 0$. Show that $E = (m/2)|x'|^2 + U(x)$ is decreasing along the trajectory.

99. Assume that $mx'' = -DU(x) - \gamma x'$, with $\gamma > 0$. Show that if $\sup_{t>0} |x| < \infty$, then x converges to a local minimum of U.

100. The law of gravity for a rocket which moves on a line through the center of the planet attracting it is given by

$$m(x + R)'' = -\frac{mgR^2}{(x + R)^2}$$

where x is the distance from the rocket to the surface of the planet and R is its radius.

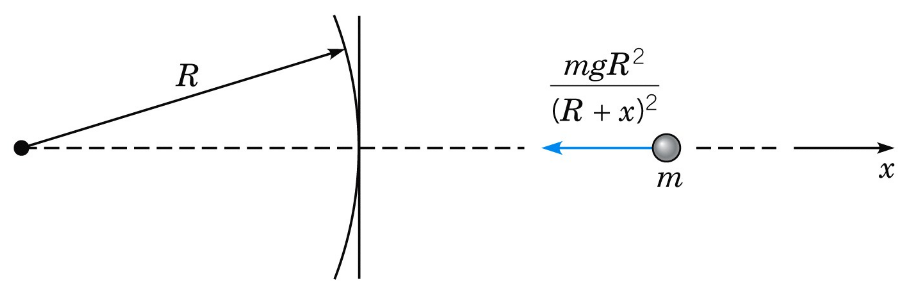


Figure 2.3.4

If initially $x(0) = R$ and $x'(0) = v_0$, determine:

(1) The maximum distance d_{max} reached by the rocket with respect to the planet.
(2) The escape velocity, i.e. the smallest value of v_0 for which $d_{max} = \infty$*.

101. (Arnold - p. 19) Draw the phase curves for the equation of an ideal planar pendulum

$$x'' = -\sin x$$

*For the earth the escape velocity is $\sim 11.2 km/sec$

102. (Arnold - p. 19) Draw the phase curves for the equation of a pendulum on a rotating axis
$$x'' = -\sin x + M$$

103. A pendulum with length l has potential energy $U(\theta) = mgl(1-\cos\theta)$ where θ is the angle that the mass makes with the vertical direction

PICTURE

(1) Show that the period of the motion starting from $\theta = \theta_0$ at rest is given by
$$T = \sqrt{\frac{8l}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

(2) Use a Taylor approximation to get that
$$T = T_0\left(1 + \frac{\theta_0^2}{16} + O(\theta_0^4)\right), \qquad T_0 = 2\pi\sqrt{\frac{l}{g}} \text{ the period of the linearized pendulum}$$

104. Let $K \in \mathbb{R}^{d\times d}$. Show that $f(x) = -Kx$ is conservative if and only if K is symmetric and find a potential for it.

105. Use Poincaré Lemma to show that every central field in $\mathbb{R}^d$ is conservative*.

106. Let $f = Du^\perp = (\partial_y u, -\partial_x u)$ defined in $\Omega \subseteq \mathbb{R}^2 \sim \mathbb{C}$ simply connected. Show that f is conservative then

(1) u is harmonic,
$$0 = \Delta u := \partial_x^2 u + \partial_y^2 u$$

(2) If $Dv = -f$ then $u + iv$ is holomorphic
$$0 = \partial_{\bar{z}}(u+iv) := \frac{1}{2}(\partial_x - i\partial_y)(u+iv) \qquad \Leftrightarrow \qquad \partial_x u = \partial_y v \text{ and } \partial_y u = -\partial_x v$$

(3) $g = Dv^\perp$ is conservative. Compute its potential.

107. Show that the field $f(x,y) = (yr^{-3}, -xr^{-3})$ with $r = \sqrt{x^2+y^2}$ and defined on $\mathbb{R}^2 \setminus \{0\}$ satisfies $\partial_x f_y = \partial_y f_x$ but it is not conservative†. Deduce from this fact that $\mathbb{R}^2\mathbb{R}^2 \setminus \{0\}$ is not simply connected.

108. Show that any trajectory of the system $x'' = 2x + y$, $y'' = x + 2y$ is bounded.‡

109. Consider Euler's method
$$v_{i+1/2} = v_{i-1/2} - (\Delta t/m)U'(x_i), \qquad x_{i+1} = x_i + \Delta t v_{i+1/2}$$

Let
$$E_{i+1/2} = \frac{m}{2}v_{i+1/2}^2 + \frac{U(x_{i+1}) + U(x_i)}{2}$$

Show that
$$\Delta E_i := E_{i+1/2} - E_{i-1/2} = \frac{1}{2}\int_{x_{i-1}}^{x_{i+1}} [U'(\xi) - U'(x_i)]d\xi$$

In particular $\Delta E_i = O(\Delta t^2)$ if U' is Lipschitz.

---

*The proof we gave before by computing the potential is actually better because it can be used also in $\mathbb{R}^d \setminus \{0\}$. Notice that for $d = 2$, $\mathbb{R}^2 \setminus \{0\}$ is not simply connected.

†Hint: Integrate f around the unit circle.

‡Hint: It is possible to solve this problem without computing the trajectories but only the potential.

110. Poincaré's Lemma Consider $\Omega \subseteq \mathbb{R}^d$ open, $f : \Omega \to \mathbb{R}^d$ smooth, and $\gamma : [0,1] \to \Omega$ smooth closed curve, i.e. $\gamma(0) = \gamma(1) = x_0$. Our goal is to show that if γ can be contracted to a point and $\partial_i f_j = \partial_j f_i$ then

$$\int_\gamma f = \int_0^1 f(\gamma(t)) \cdot \gamma'(t) dt = 0$$

(1) Let $\Gamma : [0,1]^2 \to \Omega$ be a contraction of γ. $\Gamma(t,0) = \gamma(t)$, $\Gamma(0,s) = \Gamma(1,s) = \Gamma(t,1) = x_0$ at least C^2 regular. Show that

$$\int_\gamma f = \sum_{i=1}^d \int_0^1 f_i(\Gamma(t,0)) \partial_t \Gamma_i(t,0) dt$$

(2) Show that

$$\begin{aligned} f_i(\Gamma(t,0))\partial_t\Gamma_i(t,0) &= -\int_0^1 \partial_s \left(f_i(\Gamma(t,s))\partial_t \Gamma_i(t,s) \right) ds \\ &= -\int_0^1 \partial_t \left(f_i(\Gamma(t,s))\partial_s \Gamma_i(t,s) \right) ds \end{aligned}$$

(3) Show that

$$\int_\gamma f = -\sum_{i=1}^d \int_0^1 \left(\int_0^1 \partial_t \left(f_i(\Gamma(t,s))\partial_s\Gamma_i(t,s) \right) dt \right) ds = 0$$

(4) Given $x_0 \in \Omega$ fixed and $x \in \Omega$ let $\gamma : [0,1] \to \Omega$ any curve from $\gamma(0) = x_0$ to $\gamma(1) = x$. Show that

$$U(x) = -\int_\gamma f$$

is independent of γ and $-DU = f$.

3. Linear equations with constant coefficients

Recommended lectures:

- Scalar equations:
  - Chapters 21 to 25 from The Feynman Lectures on Physics, volume I, [7].
  - Chapter 2 from Coddington's book [5].
- Systems
  - Chapter 3 from Arnold's ODE book [4].
  - Chapter 5 from Arnold's mechanics book [2].
  - Chapter I from Courant's book [6].
  - (Rotations) Sections 2, 26, and 27 from Arnold's mechanics book [2].

3.1. **Scalar equations 107.** A linear equations with constant coefficients has the form

$$a_n y^{(n)} + \ldots + a_1 y + a_0 = f(t) \qquad a_n \neq 0$$

We also write it as

$$P(D)y = f(t) \text{ where } D = d/dt \text{ and } P(x) = a_n x^n + \ldots + a_1 x + a_0 \in \mathbb{R}[x]$$

$P(D)$ is the *linear operator* associated with the equation and P is the *characteristic polynomial*. By linear we mean that if α, β are constants and x, y are functions, then it is straightforward to check that

$$P(D)(\alpha x + \beta y) = \alpha P(D)x + \beta P(D)y$$

In this sense the set of solutions of $P(D)y = 0$ is just the kernel of $P(D)$, a linear space which has dimension n as a consequence of the uniqueness of solution of the initial value problem

$$P(D)y = 0 \qquad (y(0), y'(0), \ldots, y^{(n-1)}(0)) = (y_0, y_1, \ldots, y_{n-1})$$

In other words, there is a one-to-one linear correspondence between the initial conditions and the homogeneous solutions.

On the other hand, the set of solutions of $P(D)y = f(t)$ is affine in the sense that it is given as a particular solution plus the kernel of $P(D)$.

The main observation of the theory is that

$$P(D)e^{rt} = P(r)e^{rt}$$

That is to say that $y = e^{rt}$ is an eigenvector of $P(D)$ with eigenvalue $P(r)$. Notice that it is natural to extend the coefficients from $\mathbb{R}$ to $\mathbb{C}$ where we can take advantage of the fundamental theorem of algebra.

3.1.1. *Homogeneous equations.* To illustrate the methods let us consider the second order equation

$$my'' + \gamma y' + ky = 0, \qquad m > 0, \gamma, k \geq 0$$

known as the harmonic oscillator with damping γ. The function y is the displacement from equilibrium of an object of mass m attached to a spring of stiffness k.

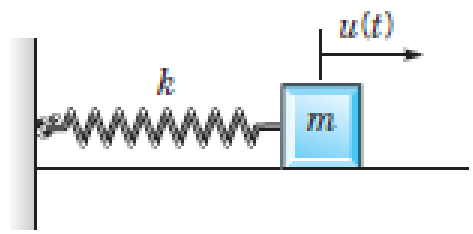


FIGURE 3.7.10 A spring–mass system.

Solutions of the form $y = e^{rt}$ are given if r is a root of the characteristic polynomial $P(x) = mx^2 + \gamma x + k$

$$r_\pm = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

We consider then three cases depending on the discriminant.

In the overdamped case $\gamma^2 > 4mk$ we have two real solutions $e^{r_\pm t}$ which are linearly independent and hence span all the solutions

$$y(t) = Ae^{r_+ t} + Be^{r_- t}$$

In the underdamped case $\gamma^2 < 4mk$ we have two complex solutions

$$e^{(-\lambda\pm i\mu)t} = e^{-\lambda t}\cos(\mu t) \pm ie^{-\lambda t}\sin(\mu t), \qquad \lambda = \frac{\gamma}{2m}, \qquad \mu = \frac{\sqrt{4mk-\gamma^2}}{2m}$$

also linear independent with respect to $\mathbb{C}$. Their real and imaginary parts give two real linear independent solutions which also span all solutions.

$$y(t) = Ae^{-\lambda t}\cos(\mu t) + Be^{-\lambda t}\sin(\mu t) = Re^{-\lambda t}\cos(\mu t - \delta) \qquad (Re^{-i\delta} = A + iB).$$

In the critically damped case $\gamma^2 = 4mk$ we only get one real solution which does not span all possible solutions. A second solution can be deduced by differentiating $P(D)e^{rt} = P(r)e^{rt}$ with respect to r

$$P(D)(te^{rt}) = P'(r)e^{rt} + P(r)te^{rt}$$

Because $r_+ = r_- = -\gamma/2m$ is a double root of P the right-hand side cancels and we recover that $te^{-\gamma/(2m)t}$ is a second solution independent of $e^{-\gamma/(2m)t}$. We get in this case the general homogeneous solution

$$y(t) = (A + Bt)e^{-\gamma/(2m)t}$$

In general to find the solutions of $P(D)y = 0$ we compute the roots of P and their multiplicities. Let us say that $r_1, \dots, r_k$ are the distinct roots and m_l is the multiplicity of r_l. Then the n complex functions given by

$$t^\alpha e^{r_\beta t} \qquad \alpha \in \{0, 1, \dots, m_\beta - 1\}, \beta \in \{1, \dots, k\}$$

are a basis for the solutions of $P(D)y = 0$. Notice also that if the initial conditions are real then the complex solution is actually real valued, thanks to the existence and uniqueness theorem.

The proofs of the previous claims are not difficult but just a standard inductive argument to show that the set of functions above is linear independent. As an easier case to try first you could look at: (1) all roots are simple, (2) there is only one root with multiplicity n. A complete proof can be found on the book by Coddington [5, Theorem 12 on Section 2.7].

A quick review of complex numbers and the complex exponential can be found in the preliminary chapter from [5] and also Chapter 22 from The Feynman Lectures on Physics, volume I.

111. Consider the equation,

$$y'' - 2by' + y = 0$$

(1) Write the characteristic polynomial and determine its roots.
(2) For which values of b are the roots are real, complex, or repeated?
(3) Write the general solution assuming that the roots are real and different.
(4) Write the general **real** valued solution assuming that the roots are complex.
(5) Write the general solution assuming that the roots are repeated.

112. Let $m, k > 0$. Solve

$$my'' + ky = 0 \qquad y(t_0) = y_0 \qquad y'(t_0) = v_0$$

113. Let $z = A + iB = Re^{i\delta} \in \mathbb{C}$ in Cartesian and polar coordinates. Use that

$$A\cos(\omega t) + B\sin(\omega t) = \mathrm{Re}(\bar{z}e^{i\omega t})$$

to show that any solution of $y'' + \omega^2 y = 0$ can be written as

$$y(t) = R\cos(\omega t - \delta)$$

114. (Berkeley Qualifying - Fall 1993) Let $x = x(t)$ be a solution of

$$x'' + 2bx' + c = 0$$

such that $x(0) = x(1) = 0$. Prove that $x(n) = 0$ for all $n \in \mathbb{Z}$.

115. Consider the system

$$x' = 2x + y \qquad y' = x + 2y$$

Find equations for $u = x + y$ and $v = x - y$ and use them to find all solutions of the system.

116. Consider $p_1, p_2, \ldots, p_n \in \mathbb{R}^d$ fixed points and $x = x(t) \in \mathbb{R}^d$ the position of a mass attached to each p_j by a spring of stiffness k_j. In other words, the mathematical model we are looking at is

$$mx'' = -\sum_{j=1}^{n} k_j(x - p_j), \qquad x = (x_1^T, \ldots, x_n^T)^T$$

Find the equilibrium and the general solution of such system.

117. Consider the two mass, three spring system without damping as in the figure
PICTURE
The equations for the displacements $x_1(t)$ and $x_2(t)$ are:

$$mx_1'' = k(x_2 - x_1) - kx_1 \qquad mx_2'' = -kx_2 - k(x_2 - x_1)$$

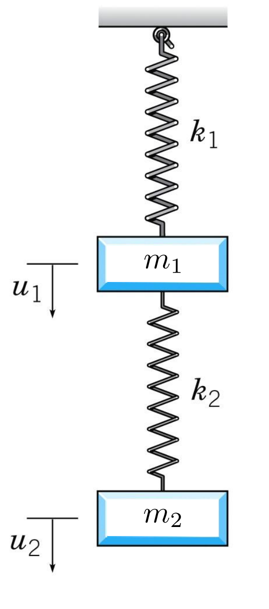
Determine equations $u = x_1 + x_2$, $v = x_1 - x_2$ and deduce from there the general solution.

118. Find all the solutions of $y^{(4)} + by'' + cy = 0$.

119. Consider the mass-spring system shown in the figure (with no gravity). The displacements u_1 and u_2 from the equilibrium positions satisfy

$$\begin{aligned} m_1 u_1'' &= -k_1 u_1 + k_2(u_2 - u_1) \\ m_2 u_2'' &= -k_2(u_2 - u_1) \end{aligned}$$

(1) Deduce a fourth order equation for u_1.
(2) Given that there is no damping, it makes sense that the general solution of the previous equation is the superposition of trigonometric functions. Prove this fact by showing that the roots of the characteristic polynomial in the equation deduced from the first part are purely imaginary.



120. Solve the system*

$$x'' + x = 2y', \qquad y'' + y = -2x'$$

121. Find all the solutions of $y''' - y = 0$.
122. Find all the solutions of $y''' + y = 0$.
123. Solve the system†

$$x' = y, \qquad y' = z, \qquad z' = x.$$

124. Find all the solutions of

$$\sum_{j=0}^{n} a^j y^{(n-j)} = 0$$

125. Find all the solutions of

$$\sum_{j=0}^{n} \binom{n}{j} a^j y^{(n-j)} = 0$$

126. Show that the kernel of D^n is the vector space of polynomials of degree $< n$.
127. Let $P \in \mathbb{R}[x]$ with $P(0) \neq 0$. Show that if all (real) solutions of $P(D)y = 0$ are bounded then P has even degree, moreover the nontrivial coefficients of P appear next to even powers.
128. Show that if m divides n then any solution of $y^{(m)} = y$ is a solution of $y^{(n)} = y$.
129. Show that if P_1 divides P_2 then any solution of $P_1(D)y = 0$ is a solution of $P_2(D)y = 0$.
130. Let $P \in \mathbb{C}[x]$. Show that

$$P(D)(t^m e^{rt}) = \sum_{j=0}^{m} \binom{m}{j} P^{(j)}(r) t^{m-j} e^{rt}$$

*Hint: $z = x + iy$
†Hint: x satisfies a third order equation.

Deduce from this identity that if r_0 is a root of P with multiplicity m_0 then $t^j e^{r_0 t} \in \operatorname{Ker} P(D)$ for $j \in \{0, 1, \ldots, m_0 - 1\}$*.

131. Let $r_0, \varepsilon \in \mathbb{R}$ and $P_\varepsilon(x) = (x - r_0 + \varepsilon)(x - r_0 - \varepsilon)$.

(1) For $\varepsilon > 0$ compute y_ε the solution of $P_\varepsilon(D)y = 0$ subject to $y(0) = 0$, and $y'(0) = 1$.

(2) Compute $y_0 = \lim_{\varepsilon \to 0} y_\varepsilon$.

(3) Check that y_0 is the solution of $P_0(D)y = 0$ subject to $y(0) = 0$, and $y'(0) = 1$.

132. Let $P \in \mathbb{R}[x]$

(1) Show that $\overline{P(z)} = P(\bar{z})$. In particular $r \in \mathbb{C}$ is a root of P of multiplicity m if and only if $\bar{r}$ is also a root of P of multiplicity m.

(2) Show that

$$\overline{P(D)y} = P(D)\bar{y}, \qquad \operatorname{Re}(P(D)y) = P(D)\operatorname{Re}(y), \qquad \operatorname{Im}(P(D)y) = P(D)\operatorname{Im}(y)$$

(3) Let $r = a + ib \in \mathbb{C}$ a root of P with multiplicity m. Show that

$$\{t^l e^{at}\cos(bt), t^l e^{at}\sin(bt)\}_{l=0}^{m-1}$$

are linear independent homogeneous solutions.

133. Show that

$$D^n(e^{rt}y) = e^{rt}(D + r)^n y$$

Deduce from this identity the formula

$$P(D)(e^{rt}y) = e^{rt}P(D + r)y$$

134. (Berkeley Qualifying - Spring 1987) Let V be a finite dimensional subspace of $C^\infty(\mathbb{R})$ that is closed under D. Prove that there exists some $P \in \mathbb{R}[x]$ such that $V = \operatorname{Ker} P(D)$.

135. Find all the real solutions of Euler's equation in the domain $r > 0$†

$$r^2u'' - ru' + 2u = 0$$

136. Find all the real solutions of Euler's equation in the domain $r > 0$

$$r^2u'' - ru' + u = 0$$

137. (Berkeley Qualifying - Fall 1995) Determine all real numbers $L > 1$ such that the boundary value problem

$$r^2u'' + u = 0, \qquad r \in [1, L], \qquad u(1) = u(L) = 0$$

has a nonzero solution.

138. Find an operator of the form $L = r^nD^n + a_{n-1}r^{n-1}D^{n-1} + \ldots + a_1 rD + a_0$ such that $u(r) = (\ln r)^2$ is a solution of $Lu = 0$.

139. Find a close formula for the Fibonacci sequence‡

$$f_{n+1} = f_n + f_{n-1}, \qquad f_0 = 0, \qquad f_1 = 1.$$

---

*Hint: Differentiate $P(D)(e^{rt}) = P(r)e^{rt}$ with respect to r.

†Hint: For $P(x) = x(x-1) + ax + b$, $D = d/dr$, and $L = r^2D^2 + \alpha rD + \beta$ we have that $Lr^\alpha = P(\alpha)r^\alpha$.

‡Hint: $f_n = Ar_+^n + Br_-^n$

140. Let $y_1 = a$, $y_2 = b$, and for $n \geq 1$, y_{n+2} is determined by the recurrence relation

$$y_{n+2} = \frac{y_{n+1} + y_n}{2}$$

Compute the limit of y_n as n goes to infinity.

141. Find a close formula for all the real solutions of the second order recurrence

$$y_{n+1} = y_n - y_{n-1}$$

142. Find a close formula for

$$y_{n+1} = 2y_n - y_{n-1}, \qquad y_0 = 0, \qquad y_{10} = 10$$

143. (Chebyshev polynomials) Let $T_0(x) = 1$, $T_1(x) = x$, and for $n \geq 1$, $T_{n+1}(x)$ is determined by the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

Compute a closed formula for $T_n(x)$ in terms of x and n.

144. Let $P_0(x) = 2$, $P_1(x) = x$, and for $n \geq 1$, $P_{n+1}(x)$ is determined by the recurrence relation

$$P_{n+1}(x) = xP_n(x) - P_{n-1}(x)$$

(1) Show that

$$P_n\left(x + x^{-1}\right) = x^n + x^{-n}$$

(2) Compute a closed formula for $P_n(x)$ in terms of x and n.

(3) Show that $P_n(2\cos\theta) = 2\cos(n\theta)$. In particular, $P_n(2x) = 2T_n(x)$ where T_n is the Chebyshev polynomial of degree n.

3.1.2. *Undetermined coefficients 141.* The goal is to find a particular solution for

$$P(D)y = f(t) \in \mathbb{C}[t]$$

Given that derivatives of polynomials are still polynomials, it is not difficult to guess that there should be a polynomial solution. Actually, if $P(0) \neq 0$, it is easy to check that there exists a particular solution $Y \in \mathbb{C}[t]$ with the same degree as f. The coefficients of Y are computed by plugging Y into the equation and matching the coefficients on both sides.

145. Find a particular solution for

$$y^{(10)} + y = 3t^2 - 1$$

146. Find a particular solution for*

$$y'' + 2y' = t^2 + 1$$

147. Let $P \in \mathbb{R}[x]$ and $f \in \mathbb{R}[t]$. Show that if 0 is a root of P with multiplicity m then $P(D)y = f(t)$ has a particular solution $Y = t^m U$ where $U \in \mathbb{C}[t]$ with the same degree as f.

148. Let $P \in \mathbb{C}[x]$. Show that if $P(r_0) \neq 0$ then

$$Y = \frac{e^{r_0 t}}{P(r_0)} \text{ is a particular solution for } P(D)y = e^{r_0 t}$$

149. Find a particular solution for

$$2y'' + 3y' + 2y = e^{-t}$$

*Hint: $u = y'$

150. Given $P \in \mathbb{R}[x]$ and $r_0 \in \mathbb{C}$ such that $P(r_0) \neq 0$ show that

$$Y = \text{Re}\left(\frac{e^{r_0 t}}{P(r_0)}\right) \text{ is a particular solution for } P(D)y = \text{Re}(e^{r_0 t})$$

151. Find a particular solution for

$$y'' + 4y = \sin t$$

152. Let $P \in \mathbb{C}[x]$. Show that if r_0 is a root of P with multiplicity m_0 then

$$Y = \frac{t^m e^{r_0 t}}{P^{(m)}(r_0)} \text{ is a particular solution for } P(D)y = e^{r_0 t}$$

153. Find a particular solution for

$$2y'' + 3y' + y = e^{-t}$$

154. Consider the following initial value problem for $m, k, \omega, F_0 > 0$

$$mu'' + ku = F_0 \cos(\omega t), \qquad u(0) = u'(0) = 0$$

(1) Compute the solution if $m\omega^2 \neq k$.
(2) (Resonance) Compute the solution if $m\omega^2 = k$.
(3) (Beat) Sketch the graph of the solution if $m\omega^2 \neq k$ but is very close to k*.
(4) Prove that the non-resonant solution converges to the resonant one as $\omega \to \sqrt{m/k}$.

155. Consider the following equation for $m, \gamma, k, \omega, F_0 > 0$

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t)$$

Notice that the solutions are given by homogenous solutions that die out as $t \to \infty$ (because $\gamma > 0$) and a steady state solution which is periodic with frequency ω. Determine the frequency ω that maximizes the amplitude of the steady state solution.

156. Let $P \in \mathbb{C}[x]$. Show that if U is a particular solution of $P(D)u = f(t)$ then

$$Y = e^{rt}U \text{ is a particular solution of } P(D + r)y = e^{rt}f(t)$$

157. Let $P \in \mathbb{C}[x]$, $r \in \mathbb{C}$, and $m \in \mathbb{N}_0$ the multiplicity of r with respect to P ($m = 0$ if $P(r) \neq 0$). Show that given $f \in \mathbb{C}[t]$ there is a unique $Q \in \mathbb{C}[t]$ with $\deg Q = \deg f$ such that $P(D)(e^{rt}t^m Q) = e^{rt}f(t)$.

158. Consider the nonhomogeneous problem for $\omega > 0$

$$y'' + 2y' + 5y = (t^2 - 1)e^{-t}\cos(\omega t)$$

(1) Compute a particular solution if $\omega \neq 2$.
(2) Compute a particular solution if $\omega = 2$.

159. Implement the following program:

- Input: $n \in \mathbb{N}$, $(a_0, a_1, \ldots, a_{n-1}) \in \mathbb{C}^n$, $r \in \mathbb{C}$, $d \in \mathbb{N}$, and $(b_0, b_1, \ldots, b_{d-1}) \in \mathbb{C}^d$.
- Output: $m \in \mathbb{N}_0$ and $(c_0, c_1, \ldots, c_d) \in \mathbb{C}^{d+1}$ such that:
  - m is the multiplicity of r with respect to the monic polynomial

$$P(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + x^n$$

($m = 0$ if $P(r) \neq 0$).

*Hint: $\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin\alpha\sin\beta$

$$- \; Y = e^{rt}t^m(c_0 + c_1t + \ldots + c_dt^d) \text{ is a particular solution of}$$
$$P(D)y = e^{rt}(b_0 + b_1t + \ldots + b_{d-1}t^{d-1} + t^d)$$

160. Show that each one of the following equations admits a (non-trivial) polynomial solution for $n \in \mathbb{N}$
(1) Hermite: $u'' - 2ru' + 2nu = 0$
(2) Laguerre: $ru'' + (1-r)u' + nu = 0$
(3) Chebyshev: $(1-r^2)u'' - ru' + n^2y = 0$
(4) Legendre: $(1-r^2)y'' - 2ru' + n(n+1)u = 0$

161. Consider Bessel's equation of order $\alpha \geq 0$
$$r^2u'' + ru' + (r^2 - \alpha^2)u = 0$$
(1) If $u(r) = r^\beta + r^\beta v(r)$ is a solution with $v(0) = 0$, show that $\beta = \pm\alpha$ necessarily.
(2) Assume that the equation admits an absolutely convergent power series solution of the form
$$J_\alpha(r) = r^\alpha \sum_{j=0}^{\infty} A_j(\alpha)r^j, \qquad A_0(\alpha) = 1$$
Compute a recurrence relation for $A_j(\alpha)$ and then deduce a close formula for it*.
(3) Show that the power series defined by this recurrence relation is absolutely convergent and therefore our ansatz satisfies the differential equation.
(4) Compute $J_{1/2}$.

3.1.3. *The convolution formula 158.* Let
$$P(x) = a_0 + a_1x + \ldots + a_nx^n \in \mathbb{C}[x], \qquad a_n \neq 0.$$
In general a particular solution of $P(D)y = f(t)$ can be computed by the following formula
$$Y(t) = (f * K)(t) = \int_{t_0}^{t} f(s)K(t-s)ds$$
where K is known as the fundamental solution and can be found as the solution of
$$P(D)y = 0, \qquad y^{(n-1)}(0) = 1/a_n, \qquad y^{(j)}(0) = 0 \text{ for } j \in \{0, 1, \ldots, n-2\}$$
Recall that the convolution has the following identities (or just prove them as an exercise):
(1) Commutativity: $f * g = g * f$.
(2) Linearity: For α, β constants, $f * (\alpha g + \beta h) = \alpha(f * g) + \beta(f * h)$.
(3) Associativity: $f * (g * h) = (f * g) * h$.
(4) Derivative: $(f * g)'(t) = (f * g')(t) + f(t)g(t_0)$.

In particular notice that the property for the derivative becomes very useful to check that $Y = f * K$ is the solution of the initial value problem
$$P(D)y = f(t), \qquad y^{(j)}(t_0) = 0 \text{ for } j \in \{0, 1, \ldots, \deg(P) - 1\}$$
Do it as an exercise!

How do we arrive to this formula? Let $P \in \mathbb{C}[x]$, $f \in C(a,b)$ and $t_0 \in (a,b)$. Our goal is to solve
$$P(D)y = f(t), \qquad y^{(j)}(t_0) = 0 \text{ for } j \in \{0, 1, \ldots, \deg(P) - 1\}$$

*Hint: Prove that $A_1(\alpha) = 0$ necessarily. Also, the identity $\Gamma(z+1) = z\Gamma(z)$ might be useful.

Given $T \in (a,b)$, consider a uniform partition of the interval from t_0 to T and then discretize f (assume without loss of generality $T > t_0$)

$$n \in \mathbb{N}, \qquad \Delta t = \frac{T - t_0}{n}, \qquad t_j = t_0 + j\Delta t, \qquad f_j = f(t_j), \qquad F = \sum_{j=1}^{n} f_j \chi_j$$

where χ_j is the indicator function of the interval $[t_{j-1}, t_j)$. In terms of the Heaviside function

$$\chi_j = H(t - t_{j-1}) - H(t - t_j), \qquad H(t) = \chi_{[0,\infty)}(t) = \begin{cases} 1 \text{ if } t \geq 0, \\ 0 \text{ else} \end{cases}$$

The main idea is to approximate f by F, reducing the problem to the case when the forcing is piecewise constant.

Let Y be the solution of

$$P(D)y = 1, \qquad y^{(j)}(0) = 0 \text{ for } j \in \{0, 1, \ldots, \deg(P) - 1\}$$

(1) Deduce that the solution of
$$P(D)y = H(t - t_j), \qquad y^{(j)}(t_j) = 0 \text{ for } j \in \{0, 1, \ldots, \deg(P) - 1\}$$
is $y_j(t) = (HY)(t - t_j)$.

(2) Deduce that the solution of
$$P(D)y = \chi_j(t), \qquad y^{(j)}(t_j) = 0 \text{ for } j \in \{0, 1, \ldots, \deg(P) - 1\}$$
is $\Delta y_j = y_{j-1} - y_j$.

(3) Deduce that the solution of
$$P(D)y = F(t) = \sum_{j=1}^{n} f_j \chi_j(t), \qquad y^{(j)}(t_j) = 0 \text{ for } j \in \{0, 1, \ldots, \deg(P) - 1\}$$
is $z = \sum_{j=1}^{n} f_j \Delta y_j = \sum_{j=1}^{n} f_j (\Delta y_j / \Delta t) \Delta t$.

(4) Take the limit as $n \to \infty$ in the previous part and recover from there the desired convolution formula.

An alternative approach using the Laplace transform

Define the Laplace transform for $s \geq 0$ as

$$\mathcal{L}f(s) = \int_0^\infty f(t) e^{-st} dt$$

Assuming the integrals are absolutely convergent we get that some of the basic properties are (exercise):

(1) $\mathcal{L}$ is linear.

(2) $\mathcal{L}(y') = -y(0) + s\mathcal{L}y$. In particular, if Y is the solution of
$$P(D)y = f(t), \qquad y^{(j)}(0) = 0 \text{ for } j \in \{0, 1, \ldots, \deg(P) - 1\}$$
then
$$\mathcal{L}Y = \frac{\mathcal{L}f}{P(s)} = \mathcal{L}f \mathcal{L}K$$
where K is the solution of
$$P(D)y = 1, \qquad y^{(j)}(0) = 0 \text{ for } j \in \{0, 1, \ldots, \deg(P) - 1\}$$

(3) $\mathcal{L}(f*g) = \mathcal{L}f\mathcal{L}g$ where

$$(f*g)(t) = \int_0^t f(s)g(t-s)ds$$

In particular, if Y and K are as above then

$$\mathcal{L}Y = \mathcal{L}f\mathcal{L}K = \mathcal{L}(f*K)$$

From this last identity now it is reasonable to guess that $Y = f*K$ is the desired solution, which then we can check and conclude the desired formula.

162. Compute $(f*g)(t)$ for each one of the following cases:
(1) $f(t) = t^m$ and $g(t) = t^n$ $(m, n \in \mathbb{N})$.
(2) $f(t) = e^{at}$ and $g(t) = e^{bt}$.
(3) $f(t) = \cos(\alpha t)$ and $g(t) = \cos(\beta t)$.
(4) $f(t) = t^{\alpha-1}e^{-t}/\Gamma(\alpha)$, $g(t) = t^{\beta-1}e^{-t}/\Gamma(\beta)$ $(\alpha, \beta > 0)$.

163. Compute the general solution for each one of the following equations
(1) $y''' - y'' + y' - y = f(t)$
(2) $y^{(4)} - y = f(t)$
(3) $y''' - 3y'' + 3y' - y = f(t)$

164. Let K be the fundamental solution for the operator $P(D)$. Notice that the convolution formula can be applied even if $f \in L^1_{loc}(a,b)^*$ is only piecewise continuous.
(1) Describe the behaviour of $Y = f*K$ at the jump discontinuities of f.
(2) Find an alternative way to compute $Y = f*K$ if f is piecewise constant by solving different initial value problems over the intervals where f remains constant.

165. Compute the fundamental solution for $P(D) = D^n$.

166. Compute the fundamental solution for $P(D) = (D-r)^n$.

167. Show that if K is the fundamental solution for $P(D)$ then $e^{rt}K$ is the fundamental solution for $P(D-r)$.

168. Consider for $j \in \{1,2\}$, $P_j \in \mathbb{C}[x]$ and K_j the fundamental solution of $P_j(D)$. Prove that $K = K_1 * K_2$ gives the fundamental solution for $P(D) = P_1(D)P_2(D)$.

169. Let $r_1, r_2, \ldots, r_n$ be distinct. Prove that

$$e^{r_1 t} * e^{r_2 t} * \ldots * e^{r_n t} = \sum_{j=1}^{n} \frac{e^{r_j t}}{Z'(r_j)} \text{ where } Z(r) = (r-r_1)(r-r_2)\ldots(r-r_n).$$

170. Let $f(t)$ be periodic. Show that the strictly damped equation $(\gamma > 0)$

$$mx'' + \gamma x' + kx = f(t)$$

admits a solution with the same period as f.

171. Let P be a polynomial and f an integrable function such that $f(t) = 0$ for all $t > 1$. Prove that there exists a polynomial Q such that $(f*P)(t) = Q(t)$ for all $t > 1$.

172. Consider the non-homogeneous recurrence relation

$$y_{n+1} = \sum_{j=0}^{k} a_j y_{n-j} + f_{n+1}$$

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*In general $f \in L^1_{loc}(\Omega)$ if $f \in L^1(K)$ for every $K \Subset \Omega$.

Deduce which conditions determine the sequence $\{k_n\}$ such that

$$Y_n = (f * k)_n = \sum_{j=0}^{n} f_j k_{n-j}$$

is a particular solution for any sequence $\{f_n\}$.

173. Compute the sequence $x * y$ for each one of the following cases:

(1) $x_n = a^n$, $y_n = b^n$.
(2) $x_n = \cos(n\alpha)$, $y_n = \cos(n\beta)$.
(3) $x_n = a^n/n!$, $y_n = b^n/n!$.
(4) For some f, g real analytic around zero, $x_n = f^{(n)}(0)/n!$, $y_n = g^{(n)}(0)/n!$.
(5) Given $n, m \in \mathbb{N}$,

$$x_k = \begin{cases} \binom{m}{k} \text{ if } k \in [0, m] \\ 0 \text{ else} \end{cases} \qquad y_k = \begin{cases} \binom{n}{k} \text{ if } k \in [0, n] \\ 0 \text{ else} \end{cases}$$

(6) $x_k = f_k$ (Fibonacci with $f_0 = f_1 = 1$) and $y_k = \binom{n}{k}$ (if $k \in [0, k]$ and $= 0$ else).
(7) Given $n, m \in \mathbb{N}$,

$$x_k = \binom{m+k}{k} \qquad y_k = \binom{n+k}{k}$$

(8) $x_n = y_n = \binom{2n}{n}$.

174. Consider the non-homogeneous vectorial recurrence relation

$$y_{n+1} = Ay_n + f_{n+1}$$

Deduce a formula for y_n in terms of y_0, $f_0, \dots, f_n$ and powers of A.

175. Compute $\mathcal{L}f$ in the following cases:

(1) $f(t) = e^{rt}$.
(2) $f(t) = e^{\alpha t}\cos(\beta t)$.
(3) $f(t) = t^{\alpha-1}$ for $\alpha > 0$.
(4) $f(t) = t^{\alpha-1}e^{rt}$ for $\alpha > 0$.

176. Compute $\mathcal{L}f$ in terms of $\mathcal{L}g$ in the following cases:

(1) $f(t) = g(\omega t)$
(2) $f(t) = g(t)e^{rt}$.
(3) $f(t) = -tg(t)$.

3.2. **Systems.**

3.2.1. *First order systems 173.* Here we consider

$$x' = Ax + f(t) \text{ where } x, f \in \mathbb{R}^d \text{ and } A \in \mathbb{R}^{d\times d}$$

Equivalently

$$x_i' = \sum_{j=1}^{d} a_{ij}x_j + f_i(t)$$

This can be easily solve if A is upper triangular* in which case the equation for x_d can be independently solved and then plugged into the other equations. Then we repeat this procedure $x_{d-1}, \dots$ etc.

*$a_{ij} = 0$ if $i < j$

In general there always exists a basis $\{\xi_1, \dots, \xi_d\}$ such that $T = B^{-1}AB$ is upper triangular where*

$$B = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \xi_1 & \xi_2 & \dots & \xi_d \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$$

Making the change of variables $y = B^{-1}x$ and $g(t) = B^{-1}f(t)$ we get from $x' = Ax + f(t)$ that y solves an upper triangular system

$$y' = Ty + g(t).$$

An alternative way to look at the same problem by analogy with the one dimensional case is to define the exponential matrix $\mu(t) = \exp(At) \in \mathbb{R}^{d\times d}$ as the unique solution of

$$\mu' = A\mu \qquad \mu(0) = I$$

Then it is straightforward to check that

$$y(t) = \exp(A(t-t_0))y_0 + \int_{t_0}^t \exp(A(t-s))f(s)ds$$

is the solution of

$$y' = Ay + f(t), \qquad y(t_0) = y_0$$

Recall some of the useful identities of the exponential (or just prove them as an exercise):

(1) The exponential can also be defined by the following absolutely convergent power series

$$\exp(At) = \sum_{j=0}^{\infty} \frac{1}{j!}(At)^j$$

(2) If $AB = BA$, then $\exp((A+B)t) = \exp(At)\exp(Bt) = \exp(Bt)\exp(At)$. In particular

$$\exp(A(t+s)) = \exp(At)\exp(As)$$

(3)

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{pmatrix} \qquad \Rightarrow \qquad \exp(At) = \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_d t} \end{pmatrix}$$

(4) If $A = BTB^{-1}$ then $\exp(At) = B\exp(Tt)B^{-1}$. In particular

$$A = B\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{pmatrix}B^{-1} \qquad \Rightarrow \qquad \exp(At) = B\begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_d t} \end{pmatrix}B^{-1}$$

(5) Nilpotent block

$$N = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \qquad \Rightarrow \qquad \exp(Nt) = \sum_{j=0}^{d-1} \frac{t^j}{j!}N^j = \begin{pmatrix} 1 & t & \dots & \frac{t^{d-1}}{(d-1)!} \\ & 1 & \ddots & \vdots \\ & & \ddots & t \\ & & & 1 \end{pmatrix}$$

*See [1, Theorem 9.4].

(6) Jordan block

$$J = \lambda I + N = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} \quad \Rightarrow \quad \exp(Jt) = e^{\lambda t}\exp(Nt) = e^{\lambda t}\begin{pmatrix} 1 & t & \dots & \frac{t^{d-1}}{(d-1)!} \\ & 1 & \ddots & \vdots \\ & & \ddots & t \\ & & & 1 \end{pmatrix}$$

Recall that for any $A \in \mathbb{C}^{n\times n}$ there exists $B \in C^{n\times n}$ such that $B^{-1}AB$ is block diagonal where each block is a Jordan block, see [1, Theorem 9.39].

177. Let $x = x(t)$ be a solution of $x' = Ax$ subject to $x(0) = \xi$

(1) If $A\xi$ is parallel to ξ we might guess from

$$x' = Ax \quad \text{or} \quad x(dt) \sim x(0) + dtAx(0) \text{ still parallel to } \xi$$

that x remains always parallel to ξ. Is this true? Can you deduce this by just considering a scalar equation?

(2) Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Compute $x(t)$ in the following cases by noticing that $A\xi$ is parallel to ξ

$$\xi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(3) Compute $x(t)$ if

$$\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

In this case it no longer holds that $A\xi$ is parallel to ξ but we could use instead that ξ is a linear combination of vectors with this property.

(4) Is there a vector $\xi \in \mathbb{R}^2$ such that

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\xi \text{ is parallel to } \xi?$$

What if we let $\xi \in \mathbb{C}^2$?

178. Compute $\exp(At)$ in each one of the following cases:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 3 & -2 \\ -3 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{pmatrix}, \qquad A = \begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$$

179. Based on the eigenvalues of A, identify the trajectories of the system $x' = Ax$ in each one of the following cases:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

PICTURES

180. Based on the eigenvalues of A, identify the trajectories of the system $x' = Ax$ in each one of the following cases:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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181. Let

$$A_{a,b} = \begin{pmatrix} a & b \\ b & a \end{pmatrix} = aI + bJ, \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(1) Compute J^n and $\exp(Jt)$.
(2) Use that I and J commute to compute $\exp(A_{a,b}t)$.

182. Let

$$A_{a,b} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = aI + bJ, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(1) Compute J^n and $\exp(Jt)$.
(2) Use that I and J commute to compute $\exp(A_{a,b}t)$.

183. (From [11, Chapter 2]) Let $a \neq b$

(1) Show that

$$\begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & \frac{a^n - b^n}{a-b} \\ 0 & b^n \end{pmatrix}$$

(2) Show that

$$\exp \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix} = \begin{pmatrix} e^a & \frac{e^a - e^b}{a-b} \\ 0 & e^b \end{pmatrix}$$

(3) Find the corresponding formulas when $a = b$ using the limits of $\frac{x^n - b^n}{x-b}$ and $\frac{e^x - e^b}{x-b}$ as $x \to a = b$.

184. (From [11, Chapter 2]) Show that every $A \in \mathbb{R}^{2\times 2}$ with $\operatorname{tr} A = 0$ satisfies*

$$\exp A = (\cos\theta)I + \frac{\sin\theta}{\theta}A, \qquad \theta = \sqrt{\det A}$$

Note: $\sin\theta/\theta = 1$ for $\theta = 0$.

185. (Berkeley Qualifying - Spring 1989) Solve $X' = AXB$ where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

186. Trigonalize A and compute $\exp(At)$ in each one of the following cases

$$A = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

*Hint: $A^2 + \det A I = 0$

187. Given $A \in \mathbb{C}^{n\times n}$ and $\lambda \in \mathbb{C}$ one of its eigenvalues, we say that η is a generalized eigenvector of order k if

$$(A-\lambda I)^{k+1}\eta = 0 \qquad \text{and} \qquad (A-\lambda I)^k \eta \neq 0.$$

(1) Let η be a generalized eigenvector of order 1. Verify that $\xi = (A-\lambda I)\eta$ is an eigenvector of A.

(2) Let η and ξ as in the previous part. Verify that the following is a solution of the system $x' = Ax$,

$$x(t) = e^{\lambda t}\eta + te^{\lambda t}\xi$$

(3) Determine the solution of the system $x' = Ax$ such that $x(0) = \eta$ is a generalized eigenvector of order k.

188. Prove that

$$\exp A = \lim_{n\to\infty}\left(I + \frac{1}{n}A\right)^n$$

189. Prove that $\det \exp A = e^{\operatorname{tr} A}$*.

190. Suppose that the eigenvalues of A are purely imaginary. Prove that the flow $\phi_t(x) = \exp(At)x$ preserves the volume.

191. Consider the n by n initial value problem

$$x' = Ax + f(t), \qquad x(0) = 0$$

(1) Implement Euler's method in the interval $[0,T]$ with step size $\Delta t = T/m$ to obtain that $x_j \sim x(j\Delta t)$ should satisfy the recursion:

$$x_{j+1} = (I + A\Delta t)x_j + f_j\Delta t, \qquad f_j = f(j\Delta t)$$

(2) Show by induction that

$$x_j = \sum_{k=1}^{j}(I + A\Delta t)^{k-1} f_{j-k}\Delta t$$

(3) Assuming that Euler's method converges to the actual solution of the system, show that

$$\lim_{m\to\infty}\sum_{k=1}^{j}(I + A\Delta t)^{k-1} f_{j-k}\Delta t = \int_0^T \exp(A(T-t))f(t)dt$$

192. Given $A \in \mathbb{C}^{n\times n}$, our goal is to define $\cos(At), \sin(At) \in \mathbb{C}^{n\times n}$:

(1) Show that the following power series are absolutely convergent:

$$\cos(At) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}(At)^{2k}, \qquad \sin(At) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(At)^{2k+1}$$

(2) Show that if $A = BTB^{-1}$ then $\cos(At) = B\cos(Tt)B^{-1}$.

(3) Show that

$$\exp(Ait) = \cos(At) + i\sin(At)$$

(4) Show that $\mu(t) = \cos(At)$ is the unique solution of

$$\mu'' = -A^2\mu, \qquad \mu(0) = I, \qquad \mu'(0) = 0$$

---

*Hint: Use $\exp A = \lim_{n\to\infty}\left(I + \frac{1}{n}A\right)^n$ and then apply the product rule to the determinant at the identity.

(5) Prove the Pythagorean identity

$$\cos^2(At) + \sin^2(At) = I$$

(6) If $AB = BA$ then

$$\cos((A+B)t) = \cos(At)\cos(Bt) - \sin(At)\sin(Bt)$$

193. Compute $\cos(At)$ in each one of the following cases

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 3 & -2 \\ -3 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{pmatrix}, \qquad A = \begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix},$$

$$A = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

194. Let $\Omega \in \mathbb{R}^{d\times d}$. Show that the solution initial value problem

$$y'' + \Omega^2 y = f(t) \qquad y(0) = y_0, \qquad y'(0) = v_0$$

is

$$y(t) = \cos(\Omega t) y_0 + tS(\Omega t) v_0 + \int_0^t K_\Omega(t-s) f(s) ds$$

where

$$S(\Omega t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (\Omega t)^{2k}, \qquad K_\Omega(t) = tS(\Omega t)$$

In particular, if Ω is invertible $S(\Omega t) = (\Omega t)^{-1} \sin(\Omega t)$ for $t \neq 0$.

195. (Floquet's Theorem) Let $A(t) \in \mathbb{R}^{d\times d}$ be continuous and periodic with period $T > 0$. Show that the solution of

$$\Phi' = A(t)\Phi, \qquad \Phi(0) = I$$

can be written as $\Phi(t) = Q(t)\exp(Bt)$ for some $B, Q(t) \in \mathbb{C}^{d\times d}$ with $Q(t)$ differentiable and periodic with period T*.

196. In this problem we will index vectors, rows, and columns starting from 0. Given $c = (c_0, \ldots, c_{n-1}) \in \mathbb{R}^n$, we define the circulant matrix $A[c] \in \mathbb{R}^{n\times n}$ as the one for which its rows are cyclically shifted version of c. For instance,

$$A[c_0, c_1, c_2] = \begin{pmatrix} c_0 & c_1 & c_2 \\ c_2 & c_0 & c_1 \\ c_1 & c_2 & c_0 \end{pmatrix}.$$

To be precise the entry in the j^{th} row and k^{th} column of $A[c]$ is c_l where $l = k - j$ modulo n. They appear in the discretization of some partial differential equations with periodic boundary conditions.

*Hint: Reduce it to the easier case when $\Phi(T) = I$.

(1) Let $J = A[0, 1, 0 \dots, 0]$. Show that

$$A[c_0, \dots, c_{n-1}] = P(J) \text{ where } P(x) = \sum_{k=0}^{n-1} c_k x^k$$

(2) Show that J is a primitive n^{th}-root of unity in $\mathbb{R}^{n\times n}$. That is to say that $k = n$ is the smallest positive power such that $J^k = I$.
(3) Find the greatest k such that $I, J, \dots, J^{k-1}$ are linearly independent.
(4) Compute the eigenvalues and eigenvectors of

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

(5) Let $\omega = e^{-2\pi i/n}$ (a primitive n^{th}-root of unity in $\mathbb{C}$). Check that

$$J = F^{-1}\,\mathrm{diag}(1, \omega, \dots, \omega^{n-1})F \qquad \text{where} \qquad F_{jk} = (\omega^{jk})$$

The map $x = (x_0, \dots, x_{n-1}) \mapsto Fx = (X_0, \dots, X_{n-1})$ such that

$$X_j = \sum_{k=0}^{n-1} x_k e^{-jk2\pi i/n}$$

is known as the discrete Fourier transform.
(6) Check that the inverse of the discrete Fourier transform gets computed by

$$x_j = \frac{1}{n}\sum_{k=0}^{n-1} X_k e^{jk2\pi i/n}$$

(7) Recall that $P(x) = \sum_{k=0}^{n-1} c_k x^k$. Show that

$$\exp A[c_0, \dots, c_{n-1}] = F^{-1}\,\mathrm{diag}(e^{P(1)}, e^{P(\omega)}, \dots, e^{P(\omega^{n-1})})F$$

197. Consider the transport equation with periodic boundary conditions

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \qquad u(x+1, t) = u(x, t)$$

For $n \in \mathbb{N}$ consider the partition of step size $\Delta x = 1/n$ and let $u_j \sim u(j\Delta x, t)$ such that

$$u_j' = \frac{u_{j+1} - u_{j-1}}{2\Delta x}, \qquad u_{j+n} = u_j$$

Find the general solution of the previous system and analyze the behavior as $n \to \infty$.

198. Consider the heat equation with periodic boundary conditions

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u(x+1, t) = u(x, t)$$

For $n \in \mathbb{N}$ consider the partition of step size $\Delta x = 1/n$ and let $u_j \sim u(j\Delta x, t)$ such that

$$u_j' = \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}, \qquad u_{j+n} = u_j$$

Find the general solution of the previous system and analyze the behavior as $n \to \infty$.

199. Consider the wave equation with periodic boundary conditions

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad u(x+1,t) = u(x,t)$$

For $n \in \mathbb{N}$ consider the partition of step size $\Delta x = 1/n$ and let $u_j \sim u(j\Delta x, t)$ such that

$$u_j'' = \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}, \qquad u_{j+n} = u_j$$

Find the general of the previous system and analyze the behavior as $n \to \infty$

200. Consider the Fibonacci sequence

$$f_{n+1} = f_n + f_{n-1}, \qquad f_0 = 0, \qquad f_1 = 1$$

(1) Show that

$$\begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = B \begin{pmatrix} r_+^n & 0 \\ 0 & r_-^n \end{pmatrix} B^{-1} \qquad r_\pm = \frac{1 \pm \sqrt{5}}{2}, \qquad B = \begin{pmatrix} 1 & 1 \\ r_+ & r_- \end{pmatrix}$$

(2) Show that $f_n^2 - f_{n-1}f_{n+1} = (-1)^n$.

(3) Show that $f_{m+n} = f_m f_{n-1} + f_{m+1} f_n$.

(4) Show that if m divides n then f_m divides f_n.

(5) Show that $f_{gcd(m,n)} = gcd(f_m, f_n)$.

201. (IMO 1979 - 6) Let A and E be opposite vertices of an octagon. A frog starts at vertex A. From any vertex except E it jumps to one of the two adjacent vertices. When it reaches E it stops. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that:

$$a_{2n-1} = 0, \qquad a_{2n} = \frac{(2+\sqrt{2})^{n-1}}{\sqrt{2}} - \frac{(2-\sqrt{2})^{n-1}}{\sqrt{2}}.$$

202. Consider the Chebyshev polynomials of the first and second kind

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad T_0(x) = 1, \qquad T_1(x) = x$$
$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \qquad U_0(x) = 1, \qquad U_1(x) = 2x$$

(1) Show that

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}$$

(2) Show that

$$\begin{pmatrix} T_n(x) \\ T_{n+1}(x) \end{pmatrix} = V^n \begin{pmatrix} 1 \\ x \end{pmatrix}, \qquad V^n = \begin{pmatrix} 0 & 1 \\ -1 & 2x \end{pmatrix}^n = \begin{pmatrix} -U_{n-2}(x) & U_{n-1}(x) \\ -U_{n-1}(x) & U_n(x) \end{pmatrix}$$

(3) Find a closed formula for V^n.

(4) Show that $U_n^2 = U_{n+1}U_{n-1} + 1$.

(5) Show that $U_{m+n} = U_m U_n - U_{m-1}U_{n-1}$.

(6) Show that

$$\begin{pmatrix} T_{n+1}(x) \\ U_n(x) \end{pmatrix} = \begin{pmatrix} x & x^2 - 1 \\ 1 & x \end{pmatrix} \begin{pmatrix} T_n(x) \\ U_{n-1}(x) \end{pmatrix}$$

(7) Compute the determinant of

$$\begin{pmatrix} 2x & 1 & & & \\ 1 & 2x & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2x & 1 \\ & & & 1 & 2x \end{pmatrix} \in \mathbb{R}^{n\times n}$$

(8) Compute the eigenvalues and eigenvectors of

$$\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix} \in \mathbb{R}^{n\times n}$$

203. An RLC circuit is modeled by a directed graph $G = (N, E)$ where for each $n \in N$ there is some voltage $v_n = v_n(t)$ and for each $e \in E$ there is a current $i_e = i_e(t)$ such that the following conservation law holds for each $n \in N$

$$\operatorname{div} i_n := \sum_{e\in E_+(n)} i_e - \sum_{e\in E_-(n)} i_e = 0 \qquad \text{(Kirchhoff)}$$

where $E_+(n)$ is the set of edges that flow into n and $E_-(n)$ is the set of edges that flow out of n.

Each edge $e \in E$ flowing from $n_-(e) \in N$ into $n_+(e) \in N$ has associated one of the following devices that help to determine the current i_e and the voltage drop $Dv_e := v_{n_+(e)} - v_{n_-(e)}$

(1) Resistor (Ohm's law)

$$R_e i_e = Dv_e$$

(2) Inductor (Faraday's law)

$$L_e i_e' = Dv_e$$

(3) Capacitor

$$C_e Dv_e' = i_e$$

In this problem I want you to implement a program that given the circuit returns the system of differential equations for $\{(i_e, \Delta v_e)\}_{e\in E}$. To be precise consider $E = E_R \cup E_L \cup E_C$ the disjoint union of edges according to the device they have and let

$$x = \begin{pmatrix} i_R \\ i_L \\ i_C \\ v \end{pmatrix} \in \mathbb{R}^{D_1}, \qquad D_1 = |E_R| + |E_L| + |E_C| + |N|$$

the variable vector with the currents and voltage. Also

$$Dx = \begin{pmatrix} i_R \\ i_L \\ i_C \\ Dv_R \\ Dv_L \\ Dv_C \end{pmatrix} \in \mathbb{R}^{D_2}, \qquad D_2 = |E_R| + |E_L| + |E_C| + |E_R| + |E_L| + |E_C|$$

We distinguish two set of equations: The algebraic ones

$$\operatorname{div} i_n = 0 \text{ for } n \in N, \quad R_e i_e = Dv_e \text{ for } e \in E_R$$

which is equivalent to

$$ADx = 0 \text{ for } A \in \mathbb{R}^{(|N|+|E_R|)\times D_2}$$

and the differential ones:

$$L_e i_e' = Dv_e \text{ for } e \in E_L, \quad CDv_e' = i_e \text{ for } e \in E_C$$

which is equivalent to

$$Dx' = BDx \text{ for } B \in \mathbb{R}^{(|E_L|+|E_C|)\times D_2}$$

Solutions of the combined system live in $V \subseteq \operatorname{Ker} AD$, the maximum subspace such that $BDV \subseteq DV$ (fill the required proofs for this claim).

- Input:
  (1) $E_R \in \mathbb{R}^{|N|\times|N|}$ the resistance matrix. $(E_R)_{ij} \geq 0$ is the resistance on an edge from i to j.
  (2) $E_L \in \mathbb{R}^{|N|\times|N|}$ the inductance matrix defined similarly as above.
  (3) $E_R \in \mathbb{R}^{|N|\times|N|}$ the capacitance matrix defined similarly as above.
- Output:
  (1) A basis $\xi_1, \dots, \xi_k \in \mathbb{R}^{D_1}$ for V.
  (2) A matrix $C \in \mathbb{R}^{k\times k}$ such that if $x = \sum_{j=1}^k y_j \xi_j$ with $y_j = y_j(t)$, then the differential system is equivalent to $y' = Cy$.

3.2.2. *Quadratic potentials 199.* Here we consider second order problems of the form

$$Mx'' + Kx = 0, \qquad \det M \neq 0$$

Keep in mind that they are actually contained in the theory of first order equations via the equivalent formulation in the phase space

$$\begin{pmatrix} x \\ v \end{pmatrix}' = \begin{pmatrix} 0 & I \\ -M^{-1}K & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}$$

A case naturally arising in mechanics is the one where M and K are symmetric and non-negative definite quadratic forms.

Recall that from the symmetry of K there exists a potential U such that $-DU(x) = -Kx$. In fact we take $U(x) = \frac{1}{2} x \cdot Kx$. Some basic properties about symmetric quadratic forms:

(1) There is an orthogonal basis of eigenvectors of K with real eigenvalues $\lambda_1 \leq \dots \leq \lambda_d$

(2) Courant's min-max identity

$$\lambda_j = \min_{\substack{V \text{ subspace of } \mathbb{R}^d \\ \dim V = j}} \max_{x \in V\setminus\{0\}} \frac{x \cdot Kx}{|x|^2}$$

See problem 209, [2, Chapter 5], or [6, Chapter I].

(3) $K \geq 0$ implies $\lambda_j = `\omega_j^2 \geq 0$.

(4) $E := \{x \cdot Kx = 1\}$ is a quadric hypersurface with axes parallel to the eigenvectors of K. Moreover, if $K \geq 0$, E is an ellipsoid with axes of length $a_1 = \omega_1^{-1} \geq \ldots \geq a_d = \omega_d^{-1} > 0$.

PICTURE

204. Consider the two mass, three spring system without damping as in the figure

PICTURE

The equations for the displacements $x_1(t)$ and $x_2(t)$ are:

$$x_1'' = -\varepsilon(x_1 - x_2) - x_1 \qquad x_2'' = -\varepsilon(x_2 - x_1) - x_2$$

(1) Determine the general solution for $x_1(t)$ and $x_2(t)$*.

(2) Compute the limits of the solutions as $\varepsilon \to 0$. Does this agree with what should be expected from physical considerations?

(3) Sketch the graph of x_1 and x_2 assuming $x_1(0) = 1$, $x_1'(0) = x_2(0) = x_2'(0) = 0$ and ε small. You should notice an alternating beat-type oscillation.

205. Consider the three-mass, four-spring system shown in the figure.

PICTURE

The displacements satisfy,

$$\begin{cases} mx_1'' = -kx_1 - k(x_1 - x_2) \\ mx_2'' = -k(x_2 - x_1) - k(x_2 - x_3) \\ mx_3'' = -k(x_3 - x_2) - kx_3. \end{cases} \qquad \Leftrightarrow \qquad x'' = \omega^2 \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} x$$

where $\omega = \sqrt{k/m}$ and $x = (x_1, x_2, x_3)^T$. Find the natural frequencies of the system, that is to say all the possible values of $\omega \geq 0$ for which there is a solution of the form $x = \xi \cos(\omega t)$.

206. A linear systems of springs with n masses has the form

$$m_i x_i'' = -\sum_{j=1}^{n_i} a_{ij}(x_i - p_j) - \sum_{j=1}^{n} b_{ij}(x_i - x_j)$$

where $m_i > 0$, $a_{ij} \geq 0$ is the stiffness of a spring between the fixed point p_{ij} and x_i, and $b_{ij} = b_{ji} \geq 0$ is the stiffness of a spring between x_i and x_j.

(1) Show that the system is conservative and compute its potential.

(2) Show that there exists at least one equilibrium x^* and that the displacement $y = x - x^*$ satisfies an equation of the form

$$m_i y_i'' = -\sum_{j=1}^{n} k_{ij} y_j \text{ for some symmetric matrix } K \geq 0.$$

Moreover, K is independent of the equilibrium chosen.

---

*Hint: Besides the straightforward procedure looking at the eigenvalues of a 4 by 4 matrix, you could also use the symmetry of the problem noticing that $x_1 \pm x_2$ satisfy a decoupled system.

(3) Determine values for the constants $\alpha_1, \dots, \alpha_n$ such that $z_i = \alpha_i y_i$ satisfies

$$z'' = -\Omega^2 z \text{ for some symmetric matrix } \Omega \geq 0.$$

In particular, the previous system can be explicitly solved after a diagonalization.

207. Let $M, K \in \mathbb{R}^{d\times d}$ symmetric such that $M > 0$ and $K \geq 0$. Consider the system $My'' = -Ky$. If $y = y(t) \in \mathbb{C}^d$ is a solution such that $y'(0)$ is parallel to $y(0)$, and $Ky(0)$ is parallel to $My'(0)$, then we might guess from $My'' = -Ky$ that y is always parallel to $y(0)$ (in the sense of complex vector spaces). Is this true?*

208. Let $M, K \in \mathbb{R}^{d\times d}$ symmetric such that $M > 0$. Prove the equivalence of the following problems:

(1)

$$\min_{x\in\mathbb{R}^d\setminus\{0\}} \frac{x\cdot Kx}{x\cdot Mx}$$

(2)

$$\min_{x\cdot Mx=1} x\cdot Kx$$

(3)

$$\min_{x\in\mathbb{R}^d} \max_{\lambda\in\mathbb{R}} x\cdot Kx - \lambda(x\cdot Mx - 1)$$

(4)

$$\max_{K\geq\lambda M} \lambda$$

209. Let $M, K \in \mathbb{R}^{d\times d}$ symmetric such that $M > 0$ and $K \geq 0$.

(1) If $y = y(t) \in \mathbb{C}^d$ is a solution such that $y'(0)$ is parallel to $y(0)$, and $Ky(0)$ is parallel to $My'(0)$, then we might guess from $My'' = -Ky$ that y is always parallel to $y(0)$ (in the sense of complex vector spaces). Is this true?†

(2) Show that if $K\xi = \lambda M\xi$ has a non trivial solution then $\lambda = \omega^2 \geq 0$‡.

(3) Show that if $\omega_1^2 \neq \omega_2^2$ are such that $K\xi = \omega_j^2 M\xi$ have non trivial solutions ξ_1 and ξ_2 respectively, then $\xi_1 \cdot M\xi_2 = \xi_1 \cdot K\xi_2 = 0$.

(4) Let $\xi \in \mathbb{C}^d$ a non trivial solution of $K\xi = \omega^2 M\xi$. Show that ξ can be completed to a basis $\zeta_1 = \xi, \zeta_2, \dots, \zeta_d$ such that

$$\zeta_i \cdot M\zeta_j = 0 \text{ for } i \neq j \qquad \Rightarrow \qquad K\zeta_i = \sum_{j=2}^{d} c_{ij} M\zeta_j \text{ for } i \geq 2$$

(5) Show that there is a basis $\xi_1, \dots, \xi_d$ such that $\xi_i \cdot M\xi_j = 0$ if $i \neq j$ and

$$K\xi_j = \omega_j^2 M\xi_j$$

for some set of frequencies $0 \leq \omega_1 \leq \ldots \leq \omega_d$. Notice that

$$\omega_j^2 = \frac{\xi_j \cdot K\xi_j}{\xi_j \cdot M\xi_j}$$

which is the analogue of $\omega^2 = k/m$ in problems with only one spring.

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*N.B. If $(K\xi - \lambda M)\xi = 0$ has non trivial solution (λ, ξ) then the pair is called an eigenvalue and an eigenvector of K with respect to M.

†N.B. If $(K\xi - \lambda M)\xi = 0$ has non trivial solution (λ, ξ) then the pair is called an eigenvalue and an eigenvector of K with respect to M.

‡$\xi \cdot K\xi = \lambda\xi \cdot M\xi$

(6) Show that

$$\omega_1^2 = \min_{y \in \mathbb{R}^d} \frac{y \cdot Ky}{y \cdot My}, \qquad \omega_d^2 = \max_{y \in \mathbb{R}^d} \frac{y \cdot Ky}{y \cdot My}$$

(7) Show that *

$$\omega_j^2 = \min_{\substack{V \text{ subspace of } \mathbb{R}^d \\ \dim V = j}} \max_{y \in V \setminus \{0\}} \frac{y \cdot Ky}{y \cdot My}$$

210. Let $M, K, \tilde{K} \in \mathbb{R}^{d \times d}$ symmetric such that $M > 0$ and $\tilde{K} \geq K \geq 0$. Let $0 \leq \omega_1 \leq \ldots \leq \omega_d$ and $0 \leq \tilde{\omega}_1 \leq \ldots \leq \tilde{\omega}_d$ the frequencies associated to the systems $My'' = -Kx$ and $My'' = -\tilde{K}y$ respectively. Show that $\omega_j \leq \tilde{\omega}_j$.

211. Consider the ellipsoid with axes $a_1 \geq \ldots \geq a_d > 0$

$$E = \left\{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d : \sum_{i=1}^{d} \left(\frac{x_i}{a_i} \right)^2 = 1 \right\}$$

Show that for any subspace V of dimension $d-1$, $\tilde{E} = E \cap V$ is an ellipsoid with axes $\tilde{a}_1 \geq \ldots \geq \tilde{a}_d > 0$ such that

$$a_{j+1} \geq \tilde{a}_{j+1} \geq a_j \geq \tilde{a}_j.$$

212. Let $M, \Gamma, K \in \mathbb{R}^{d \times d}$ symmetric and $M > 0$, $\Gamma, K \geq 0$.

(1) Show that the system $My'' + \Gamma y' + Ky = 0$ has at least one non-trivial solution of the form $y(t) = e^{rt}\xi$.

(2) Show that any non-trivial solution of the form $y(t) = e^{rt}\xi$ satisfies $\mathrm{Re}(r) \leq 0$.

(3) Show that there exists a change of coordinates such that $My'' + \Gamma y' + Ky = F(t)$ is equivalent to a system of the form

$$z_i'' + 2 \sum_{i,j=1}^{d} \tilde{\gamma}_{ij} z_j' + \omega_i^2 z_i = \tilde{f}_i(t)$$

where $\tilde{\Gamma} = (\tilde{\gamma}_{ij})$ is symmetric and ≥ 0.

(4) Show that if γ is an eigenvalue of $\tilde{\Gamma}$, then $y'' + 2\Gamma y' + \omega^2 y = 0$ has solutions of the form $y(t) = u(t)\xi$ where u is any solution of $u'' + 2\gamma u' + \omega^2 u = 0$.

3.2.3. *Rotations 208.* Here we mainly consider the equation $y' = \Omega y$ such that its corresponding flow $\phi_t(y) = \exp(\Omega t)y$ is an isometry. In other words $\exp(\Omega t) \in SO(d)$ the special orthogonal group

$$\begin{aligned} SO(d) &= \{A \in \mathbb{R}^{d \times d} : |Ax| = |x| \ \forall x \in \mathbb{R}^d, \det A = 1\} \\ &= \{A \in \mathbb{R}^{d \times d} : AA^T = I, \det A = 1\} \end{aligned}$$

We will soon discover that for $d = 3$ the cross product plays an important role. Recall that it can be defined by:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$$

*Hint: $\dim V = j$ implies $V \cap span(\xi_j, \ldots, \xi_d) \neq \{0\}$

Some of the useful identities of the cross product are (as always, prove them as an exercise):

(1) The cross product is linear.
(2) Skew-symmetry: $x \times y + y \times x = 0$.
(3) Jacobi: $x \times (y \times z) + y \times (z \times x) + z \times (x \times y) = 0$.
(4) Geometric interpretation: $\det(x, y, z) = (x \times y) \cdot z$. In particular $x \times y$ is perpendicular to x and y, and $|x \times y|$ is the area of the parallelogram

$$\{z \in \mathbb{R}^3 : z = sx + ty \text{ for } s, t \in [0, 1]\}$$

213. Let $\omega \in \mathbb{R}^3$. Solve $y' = \omega \times y$.

214. Let $\omega \in \mathbb{R}^3$. Show that the flow generated by $y' = \omega \times y$ is an isometry at each time.

215. Show that the flow generated by $y' = f(y)$ is an isometry at each time if and only if $f(y) \perp y$ for every $y \in Dom f$.

216. Show that the flow generated by $y' = \Omega y$ is an isometry at each time if and only if $\Omega + \Omega^T = 0$, that is to say that Ω is a skew-symmetric matrix.

217. Let $\omega \in \mathbb{R}^3$. Check that

$$\Omega y = \omega \times y \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

218. Let

$$E(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(1) Check that $E(t) \in SO(3)$ for every $t \in \mathbb{R}$.
(2) Describe in geometric terms the flow $\phi_t(y) = E(t)y$.
(3) Show that

$$E(t) = \exp(Zt) \text{ for } Z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(4) Describe in geometric terms the vector field $f(y) = Zy$.
(5) Let $B \in SO(3)$. Describe in geometric terms the flow $\psi_t(y) = BE(t)B^{-1}y$.
(6) Show that $B\exp(Zt)B^{-1} = \exp(\Omega t)$ for $\Omega = BZB^{-1}$.
(7) Show that Ω is skew symmetric and $\Omega\omega = 0$ if and only if ω is parallel to Be_3.

219. Given $\omega \in \mathbb{R}^3$ construct $\Omega \in \mathbb{R}^{3\times 3}$ such that the flow $\exp(\Omega t)$ gives a rotation around the axis generated by ω with angular velocity $|\omega|$. Note: There are at least two possible answers for this exercise depending on the orientation one chooses for the rotation.

220. Given $\omega \in \mathbb{R}^3$ show that $f(y) = \omega \times y$ is the unique linear vector field that generates a rotation around the axis spanned by ω with angular velocity $|\omega|$ and such that $\det(\omega, y, f(y)) \geq 0$.

221. Show that for every $\Omega \in \mathbb{R}^{3\times 3}$ skew-symmetric there exists some $\varpi \geq 0$ such that $\Omega^3 + \varpi^2\Omega = 0$. Explain in geometric terms why it is reasonable to call ϖ the angular speed of the rotation $t \mapsto \exp(\Omega t)$.

222. Let $x, y, z \in \mathbb{R}^3$ and $X, Y, Z \in \mathbb{R}^{3\times 3}$ such that

$$X = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{pmatrix}$$

Show that under this identification $Z = [X, Y] = XY - YX$ if and only if $z = x \times y$.

223. Consider a curve $x = x(t) \in \mathbb{C} \setminus \{0\}$. In polar coordinates we denote $x = re^{i\theta}$ and define $\omega = \theta'$ as the angular velocity. Show that

$$x'' = (r'' - r\omega^2)e^{i\theta} + (2r'\omega + r\omega')ie^{i\theta}$$

The terms $-r\omega^2$ and $2r'\omega$ are known by the names of centripetal and Coriolis acceleration respectively.

224. Consider a curve $x = x(t) \in \mathbb{R}^3 \setminus \{0\}$. In polar coordinates we denote $r = |x|$ and $\theta = x/r$ and define $\omega = \theta \times \theta'$ as the angular velocity:

(1) Show that θ, θ', and ω are pairwise orthogonal.
(2) Show that $\omega \times \theta = \theta'$ and give a geometric interpretation of the definition of angular velocity based on this identity.
(3) Show that $\omega' = \theta \times \theta''$.
(4) Show that

$$x'' = (r'' - r|\omega|^2)\theta + (2r'\omega + r\omega') \times \theta$$

225. Consider curves $q = q(t) \in \mathbb{R}^3$ and $B = B(t) \in SO(3)$

(1) Explain why $Q = B^{-1}q$ represents the curve $q = q(t)$ as seen by an observed that rotates according to B.
(2) Show that there exists $\Omega = \Omega(t) \in \mathbb{R}^{3\times 3}$ skew-symmetric such that

$$q' = B(Q' + \Omega Q)$$

(3) Show that

$$q'' = B(Q'' + \Omega^2 Q + 2\Omega Q' + \Omega' Q)$$

(4) Show that if q satisfies Newton's equation $mq'' = f(q, q')$ then

$$mQ'' = \underbrace{F}_{=B^{-1}f(BQ,(BQ)')} - \underbrace{m\Omega^2 Q}_{\text{centripetal force}} - \underbrace{2m\Omega Q'}_{\text{coriolis force}} - \underbrace{m\Omega' Q}_{\text{inertial force of rotation}}$$

226. Consider a curve $x = x(s) \in \mathbb{R}^3$ parametrized such that $\tau = x'$ is a unit vector for every time. Define $n = \tau'/|\tau'|$ the normal vector, and $b = \tau \times n$ the binormal vector:

(1) Show that τ, n, and b are an orthonormal frame.
(2) Show that τ, n, and b satisfy a system of equations of the form

$$\begin{pmatrix} \tau' \\ n' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \chi \\ 0 & -\chi & 0 \end{pmatrix} \begin{pmatrix} \tau \\ n \\ b \end{pmatrix}$$

The functions κ and χ are called the curvature and torsion respectively.

(3) Describe all the curves with constant curvature and torsion.
(4) Consider a curve $x = x(t) \in \mathbb{R}^3$ and let $s = s(t)$ be its arc length reparametrization such that $x = x(s)$ has unit speed, or equivalently $|ds/dt| = |dx/dt|$. Show that

$$\frac{d^2x}{dt^2} = \frac{d^2s}{dt^2}\tau + \kappa\left|\frac{ds}{dt}\right|^2 n$$

(5) Show that if $mx'' = f$ then f belongs to the plane spanned by τ and n.

227. Prove that $SO(d)$ is a group under the product of matrices. In other words, $A, B \in SO(d)$ implies that $AB, BA \in SO(d)$ and every $A \in SO(d)$ is invertible with $A^{-1} \in SO(d)$.

228. Let $\gamma : \mathbb{R} \to SO(d)$ a differentiable curve such that $\gamma(0) = I$. Prove that

$$\gamma'(0) \in so(d) = \{\Omega \in \mathbb{R}^{d\times d} : \Omega + \Omega^T = 0\} \text{ skew symmetric matrices}$$

Moreover, for every $\Omega \in so(d)$ the curve $\gamma(t) = \exp(\Omega t) \in SO(d)$ is such that $\gamma(0) = I$ and $\gamma'(0) = \Omega$.

229. Let $\Omega \in \mathbb{R}^{d\times d}$ skew-symmetric:

(1) For d odd show that there is $x \in \mathbb{R}^d \setminus \{0\}$ such that $\Omega x = 0$.

(2) For d even show that zero is not necessarily an eigenvalue of Ω.

230. Let $B \in SO(d)$:

(1) Show that for every $A \in SO(d)$ the conjugation

$$\Psi_B A = BAB^{-1} \in SO(d)$$

.

(2) Show that $\Psi_{AB} = \Psi_A \circ \Psi_B$.

(3) Show that $\Psi_B^{-1} = \Psi_{B^{-1}}$.

231. Given $B \in SO(d)$ and $\Omega \in so(d)$ define

$$Ad_B\Omega = \left.\frac{d}{dt}\right|_{t=0} \Psi_B\gamma(t)$$

where γ is any differentiable curve such that $\gamma(0) = I$ and $\gamma'(0) = \Omega$. This transformation is also known as the adjoint representation because Ad maps $B \in SO(d)$ to Ad_B which is an automorphism of $so(d)$.

(1) Show that

$$Ad_B\Omega = B\Omega B^{-1} \in so(d)$$

in particular, the definition is independent of the curve.

(2) Show that Ad_B is linear.

(3) Show that $Ad_{AB} = Ad_A \circ Ad_B$.

(4) Show that $Ad_B^{-1} = Ad_{B^{-1}}$.

(5) Show that $Ad_B[X, Y] = [Ad_B X, Ad_B Y]$.

232. Let $X, Y \in so(d)$ and define the derivation

$$ad_X Y = \left.\frac{d}{dt}\right|_{t=0} Ad_{\gamma(t)}Y$$

where γ is any differentiable curve such that $\gamma(0) = I$ and $\gamma'(0) = X$.

(1) Show that

$$ad_X Y = [X, Y] = XY - YX \in so(d)$$

in particular, the definition is independent of the curve.

(2) Show that if $[X, Y] = 0$ then

$$\exp(Xt)\exp(Ys) = \exp(Ys)\exp(Xt) = \exp(Xt + Ys)$$

(3) Show that ad is linear in both entries.

(4) Show that ad satisfies the product rule $ad_X[Y, Z] = [ad_X Y, Z] + [Y, ad_X Z]$.

(5) Show that $ad_{[X,Y]} = [ad_X, ad_Y] = ad_X ad_Y - ad_Y ad_X$.

233. Define the quaternions as

$$\mathbb{H} = \{q = (a, b) \in \mathbb{R} \times \mathbb{R}^3\}$$

with the vector space structure of the Euclidean space and the product

$$(a, b)(c, d) = (ac - b \cdot c, ad + bc + b \times c)$$

Analogous to the complex number we identify $\mathbb{R}$ with the quaternions of the form $(a, 0) = a$, $\mathbb{R}^3$ with $(0, b) = b$, and for $q = (a, b)$

$$\bar{q} = (a, -b), \qquad |q|^2 = q\bar{q} = a^2 + |b|^2$$

(1) Check that the product of quaternions is associative but not commutative.
(2) Check that $|q_1 q_2| = |q_1||q_2|$.
(3) Check that for $q \neq 0$, $q^{-1} = q/|q|^2$ satisfies $qq^{-1} = q^{-1}q = 1$ and $|q^{-1}| = |q|^{-1}$. In particular, $S^3 = \{q \in \mathbb{H} : |q| = 1\}$ is a group. Notice that the tangent space of S^3 at the identity is $\mathbb{R}^3$.
(4) Given $b \in S^3$ consider the conjugation $a \in S^3 \mapsto \Psi_b a = bab^{-1} \in S^3$. Given a differentiable curve $\gamma = \gamma(t) \in S^3$ such that $\gamma(0) = 1$ and $\gamma'(0) = x \in \mathbb{R}^3$ consider the adjoint representation

$$Ad_b x = \left.\frac{d}{dt}\right|_{t=0} \Psi_b \gamma(t) = bxb^{-1} \in \mathbb{R}^3$$

Show that Ad_b is a rotation and find its axis.
(5) Parametrize $b \in S^3$ as $b = (\cos\theta, \sin\theta\ n)$ with $n \in S^2$. Show that Ad_b is a rotation around n by an angle 2θ.

234. A rotation R can also be characterized by the so called Euler's angles φ, θ, and ψ. The idea is to rotate a orthonormal frame (u_1, u_2, u_3) to the canonical basis (e_1, e_2, e_3) in three steps. The following describes the inverse transformation:
(1) Rotate (e_1, e_2, e_3) around e_3 by an angle φ to get (n, e_2', e_3).
(2) Rotate (n, e_2', e_3) around n by an angle θ to get (n, e_2'', u_3).
(3) Rotate (n, e_2'', u_3) around u_3 by an angle ψ to get (u_1, u_2, u_3).
PICTURE
(1) Show that the matrix of the complete transformation is

$$\begin{aligned} R &= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi & \cos\psi\sin\varphi + \sin\psi\cos\theta\cos\varphi & \sin\psi\sin\theta \\ -\sin\psi\cos\varphi - \cos\psi\cos\theta\sin\varphi & -\sin\psi\sin\varphi + \cos\psi\cos\theta\cos\varphi & \cos\psi\sin\theta \\ \sin\theta\sin\varphi & -\sin\theta\cos\varphi & \cos\theta \end{pmatrix} \end{aligned}$$

(2) Determine φ, θ, and ψ in terms of α such that

$$R = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

(3) With the previous notation, show that if $(\varphi, \theta, \psi) = (\varphi(t), \theta(t), \psi(t))$ is such that $(\varphi(0), \theta(0), \psi(0)) = 0$ then

$$R'(0) = \begin{pmatrix} 0 & \varphi'(0) + \psi'(0) & 0 \\ -\varphi'(0) - \psi'(0) & 0 & \theta'(0) \\ 0 & -\theta'(0) & 0 \end{pmatrix}$$

Explain why we miss the infinitesimal rotations around e_2.

The main construction in this section comes from the flow $\phi^t : \mathbb{R}^d \to \mathbb{R}^d$ ($t \in \mathbb{R}$) generated by a vector field $X : \mathbb{R}^d \to \mathbb{R}^d$

$$\frac{d}{dt}\phi^t = X(\phi_t), \qquad \phi^0 = id$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is (backwardly) transported by ϕ^t if one considers

$$f_t = f \circ \phi^t$$

Notice that the graph of f^t follows the direction of $-V$ as t goes on.

Similarly we can also transport in a natural way other objects such as a differential form $\omega \in \Omega^k(\mathbb{R}^n)$

$$\omega_t = (\phi^t)^*\omega$$

or another vector field $Y : \mathbb{R}^n \to \mathbb{R}^n$

$$Y_t = (\phi^{-t})_* Y_{\phi^t}$$

In any of these cases we define the Lie derivative in the same way:

$$\mathcal{L}_X f = \left.\frac{d}{dt}\right|_{t=0} f_t = Xf, \qquad \mathcal{L}_V \omega = \left.\frac{d}{dt}\right|_{t=0} \omega_t, \qquad \mathcal{L}_X Y = \left.\frac{d}{dt}\right|_{t=0} Y_t$$

Some of the useful identities are:

(1)

4. Central forces 230

Recommended lectures:

- Chapter 2 from Arnold's mechanics book [2].
- Chapter 2 from Arnold's celestial mechanics book [3].

Here we consider in $\mathbb{R}^3 \setminus \{0\}$ the problem with a central force (which is always conservative)

$$mx'' = -U'(r)\theta, \qquad r = |x|, \qquad \theta = x/r$$

We take advantage of the rotational symmetry by writing the equations in polar coordinates (recall $\omega = \theta \times \theta'$)

$$mx'' = m(r'' - r|\omega|^2)\theta + m(2r'\omega \times + r\omega') \times \theta = -U'(r)\theta$$

Taking the components in the radial and spherical directions

$$m(r'' - r|\omega|^2) = -U'(r), \qquad (2r'\omega + r\omega') \times \theta = 0$$

Taking the cross product with θ in the second equation and using that $a \times (b \times c) = -(a \cdot b)c + (a \cdot c)b$ we get that $0 = 2r'\omega + r\omega' = (r^2\omega)'/r$, therefore the angular momentum

$$M := mr^2\omega = x \times mx' \text{ is conserved.}$$

Plugging $\omega = M/(mr^2)$ into the radial equation we get

$$mr'' = -U'(r) + mr|\omega|^2 = -U'(r) + \frac{|M|^2}{mr^3} = -V'(r) \qquad V(r) = U(r) + \frac{|M|^2}{2mr^2}$$

In other words, the radius evolves as in a one dimensional problem governed by the *effective potential* V. In particular

$$\frac{m}{2}(r')^2 + V(r) = h \text{ is conserved.}$$

For example, the gravitational potential $U(r) = -r^{-1}$ has associated to it an effective potential of the form $V(r) = -r^{-1} + Cr^{-2}$

PICTURE

Notice that r is periodic for $h \in [(2C)^{-1}, 0)$, and $r \to \infty$ as $|t| \to \infty$ for $h \geq 0$.

235. Let $x = r\theta \in \mathbb{R}^3$ subject to $mx'' = -U'(r)\theta$. Prove from the conservation of angular momentum that the mass moves faster when r is smaller and slower when r is large.

236. Let $x = r\theta \in \mathbb{R}^3$ subject to $mx'' = -U'(r)\theta$. Prove from the conservation of angular momentum that the trajectory is contained on a fixed plane, therefore allowing to reduce the three dimensional system to a two dimensional one.

237. Let $x = r\theta \in \mathbb{R}^3$ subject to $mx'' = -U'(r)\theta$. Prove from the conservation of angular momentum that the segment from the origin to x sweeps out equal areas in equal time (constant sectorial velocity)*.

238. Let $x = re^{i\theta} \in \mathbb{C} \sim \mathbb{R}^2$ subject to $mx'' = -U'(r)e^{i\theta}$. Explain why the following trajectory is not possible

PICTURE

239. Let $x = re^{i\theta} \in \mathbb{C} \sim \mathbb{R}^2$ subject to $mx'' = -U'(r)e^{i\theta}$.

(1) Show that $M = mr^2\theta'$ is conserved. (Notice that in the two dimensional case M is not a vector.)

(2) Show that $h = (m/2)(r')^2 + V(r)$ is conserved where $V(r) = U(r) + M^2/(2mr^2)$.

(3) Show that if θ is not constant then it is invertible and $r(\theta) = r(t(\theta))$ satisfies

$$\frac{M^2}{2m}\left(r^{-2}\frac{dr}{d\theta}\right)^2 + V(r) = h$$

In particular $\rho(\theta) = r^{-1}$ satisfies

$$\frac{M^2}{m}\rho'' = V'(\rho^{-1})\rho^{-2}$$

(4) Assuming that $\{V \leq h\} = [r_{min}, r_{max}]$, show that the orbit is closed if

$$\Phi = \frac{M}{\sqrt{2m}}\int_{r_{min}}^{r_{max}} \frac{\rho^{-2}d\rho}{\sqrt{h - V(\rho)}}$$

is commensurable with 2π. Otherwise $x = re^{i\theta}$ is everywhere dense on the annulus $r \in [r_{min}, r_{max}]$.

240. Let $x = re^{i\theta} \in \mathbb{C} \sim \mathbb{R}^2$ subject to $mx'' = -U'(r)e^{i\theta}$. As usual $M = mr^2\theta'$, $V = U + M^2/(2mr^2)$, and $h = (m/2)(r')^2 + V$. Assuming that the trajectory is a circle of radius r_0 centered at the origin:

(1) Show that V has either a local maximum or a local minimum at r_0.

(2) Show that θ' is constant.

---

*Hint: Interpret $|M|$ as the area of a region.

(3) Assuming Kepler's observation $r_0^3 = CT^2$, where $T = (2\pi r_0)/(r_0\theta')$ is the period of revolution, show that $U(r) = -Gr^{-1}$ (plus your favorite constant).

241. Let $x = re^{i\theta} \in \mathbb{C} \sim \mathbb{R}^2$ subject to $x'' = -kr^{-2}e^{i\theta}$ and assume that θ is not constant

(1) Show that $\rho(\theta) = r^{-1}$ satisfies

$$\frac{M^2}{2}(\rho')^2 + \frac{M^2}{2}\rho^2 - k\rho = h$$

Therefore (and after a convenient rotation)

$$r(\theta) = \frac{p}{1 + e\cos\theta}$$

which is the equation of a conic section with eccentricity $e \geq 0$ and with the origin as one of the foci*.

(2) Determine in terms of h in which cases the trajectory is a circle, ellipse, parabola, or hyperbola.

(3) Assume the trajectory is an ellipse with major semi-axis a and T is the period of one revolution. Prove that a^3T^{-2} is constant, as expected from Kepler's observations.

242. Let $x = re^{i\theta} \in \mathbb{C} \sim \mathbb{R}^2$ subject to $x'' = -kr^{-2}e^{i\theta}$ with the initial condition $x(0) = 1$ and $x'(0) = v_0 i$ for some $v_0 > 0$.

(1) Determine v_0 such that the trajectory of x is a circle, also known as the first escape velocity.

(2) Determine v_0 such that the trajectory of x is a parabola, also known as the second escape velocity.

243. Consider the two body problem in $\mathbb{R}^2$

$$m_1 x_1'' = -U'(r)e^{i\theta}, \qquad m_2 x_2'' = U'(r)e^{i\theta}$$

where $x_1 - x_2 = re^{i\theta}$. Show that $y = x_1 - x_2$ satisfies

$$my'' = -U'(r)e^{i\theta} \text{ where } m = \frac{m_1 m_2}{m_1 + m_2}$$

Hence if $M > 0$, the trajectory can be computed from $r = r(\theta)$

$$\frac{M^2}{2m}\left(r^{-2}\frac{dr}{d\theta}\right)^2 + V(r) = h \text{ where } V = U + \frac{|M|^2}{2mr^2}$$

244. For $U = -kr^{-1}$ describe the trajectories of the two-body problem in $\mathbb{R}^3$†.

245. Consider $mx'' = |x|^{-3}(x \times x')$ in $\mathbb{R}^3$. Show that any trajectory always remain on a cone with vertex at the origin.

5. Lagrangian Mechanics

Recommended lectures:

- Calculus of Variations:
  - Chapter 19 from The Feynman Lectures on Physics, volume II, [8].
  - Chapter 3 from Arnold's mechanics book [2].
  - The first four chapters from the book of Gelfand and Fomin [10].

*Hint: There are at least two techniques you could try: Separation of variables for the first order equation or look at a second order equation for $\rho = \rho(\theta)$.

†Hint: The center of mass satisfies an independent equation.

- Holonomic Constrains:
  - Sections 17 and 21 from Arnold's mechanics book [2].

5.1. **Calculus of Variations 241.** Consider Newton's differential equation

$$mx'' = -DU(t,x)$$

an integral formulation can be written as

$$mx'(t) = mx'(t_0) - \int_{t_0}^{t} DU(s,x(s))ds \qquad \forall\, t \in [t_0,t_1]$$

In an equivalent an more flexible way we could also integrate against a *test function*

$$\int_{\mathbb{R}} mx''(s)\varphi(s) + DU(s,x(s),x'(s))\varphi(s)ds = 0 \qquad \forall\, \varphi \in C_0^\infty([t_0,t_1])$$

the advantage is that integrating by parts we get a formulation that does not require x to have two derivatives

$$\int_{t_0}^{t_1} mx'(s)\varphi'(s) - DU(s,x(s))\varphi(s)ds = 0$$

On the other hand we recognize an exact derivative on the left-hand side

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_{t_0}^{t_1} \frac{m}{2}|x'(s)+\varepsilon\varphi'(s)|^2 - U(s,x(s)+\varepsilon\varphi(s))ds = 0$$

In other words, the solutions of the differential equation are the critical points (which in this case are curves) of the *action*

$$\Phi(x) = \int_{t_0}^{t_1} L(s,x(s),x'(s))ds, \qquad L(t,x,v) = \frac{m}{2}|v|^2 - U(t,x)$$

where L is known as the *Lagrangian*.

In general, the Euler-Lagrange equation for the critical points is

$$\frac{d}{dt}\partial_v L = \partial_x L$$

where $\partial_v L$ is known as the *generalized momentum* and $\partial_x L$ as the *generalized force*.

A typical application use of the equation above is given in optimization problems when one is asked to maximize or minimize the action.

246. A cylindrical surface in $\mathbb{R}^3$ around the e_3 axis is given by

$$(x,y,z) = (r(z)\cos\theta, r(z)\sin\theta, z)$$

Its surface area between $z=-1$ and $z=1$ is

$$S(r) = 2\pi \int_{-1}^{1} \sqrt{1+(r')^2}\; rdz$$

Minimize $S(r)$ subject to $r(-1)=r(1)=R$. Notice that there is some $R_0>0$ such that the minimal surface is well defined if and only if $R \geq R_0$.

247. Consider a Lagrangian $L = L(t,x,v) = L(t,x_1,\ldots,x_d,v_1,\ldots,v_d)$ defined in $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$

(1) Check that if L is independent of t then

$$\frac{d}{dt}\partial_v L = \partial_x L \qquad \Rightarrow \qquad x'\cdot\partial_v L - L = H \text{ is conserved}$$

(2) Check that if L is independent of x_i then

$$\frac{d}{dt}\partial_v L = \partial_x L \qquad \Rightarrow \qquad \partial_{v_i} L = p_i \text{ is conserved}$$

(3) Check that if L is rotationally invariant (i.e. $L(t, Rx, Rv) = L(t, x, v)$ for any $R \in SO(d)$) then

$$\frac{d}{dt}\partial_v L = \partial_x L \qquad \Rightarrow \qquad \Omega x \cdot \partial_v L = M_\Omega \text{ is conserved for every } \Omega \in so(d)$$

248. (Noether's Theorem) Let $\Phi_t : \mathbb{R}^d \to \mathbb{R}^d$ be the flow generated by the vector field

$$V(q) = \left.\frac{d}{dt}\right|_{t=0} \Phi_t(q)$$

We say that L admits the symmetry given by Φ if

$$L(\Phi_t(q), D\Phi_t(q)\dot{q}) = L(q, \dot{q}) \qquad \forall\ t \in \mathbb{R}$$

Show that in this case

$$\frac{d}{dt}\partial_{\dot{q}} L = \partial_q L \qquad \Rightarrow \qquad I(q, \dot{q}) = \partial_{\dot{q}} L(q, \dot{q}) \cdot V(q) \text{ is conserved.}$$

249. Let $f : \mathbb{R} \to \mathbb{R}$ smooth and defining the surface of revolution

$$S = \{(x, y) \in \mathbb{R}^n \setminus \{0\} \times \mathbb{R} : y = f(|x|^2)\}$$

(1) Compute the formula for the kinetic energy of a curve of the form

$$\tilde{\gamma} = (\gamma, f(|\gamma|^2)) \in S$$

where $\gamma : [0, 1] \to \mathbb{R}^n \setminus \{0\}$.

(2) Compute the Euler-Lagrange equation for the kinetic energy.

(3) Compute the law of conservation of energy ($\gamma' \cdot \partial_v E - E = H$).

(4) Given the rotational symmetry it follows that the Lagrangian is invariant by rotations and there is a conservation law of angular momentum ($\Omega\gamma \cdot \partial_v E = M_\Omega$ for all $\Omega \in so(n)$). Compute this conservation law and give a geometric interpretation of it.

250. Let $\varphi : \mathbb{R}^n = \{q \in \mathbb{R}^{n+1} : q_{n+1} = 0\} \to S^n = \{q \in \mathbb{R}^{n+1} : |q| = 1\}$ be the stereographic map from the north pole e_{n+1} defined as the intersection of $S^n \setminus \{e_{n+1}\}$ with the line containing q and e_{n+1}

PICTURE

(1) Show that

$$\varphi(q) = q + \frac{|q|^2 - 1}{|q|^2 + 1}(e_{n+1} - q)$$

(2) Show that for $v \in \mathbb{R}^n$

$$D\varphi(q)v = \frac{2}{|q|^2 + 1}v + \frac{4(q \cdot v)}{|q|^2 + 1}(e_{n+1} - q)$$

(3) Given $v, w \in \mathbb{R}^n$, let us define the metric at q by

$$g_q(v, w) = D\varphi(q)v \cdot D\varphi(q)w$$

Show that

$$g_q(v, w) = \left(\frac{2}{|q|^2 + 1}\right)^2 v \cdot w$$

(4) Compute the Euler-Lagrange equation of the kinetic energy

$$E(\gamma) = \frac{1}{2}\int_0^1 \left(\frac{2|\gamma'(t)|}{|\gamma(t)|^2+1}\right)^2 dt$$

(5) Compute the law of conservation of energy for E.

(6) Compute the law of conservation of angular momentum for E.

251. Let $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a Lagrangian, $x_0 = x_0(t)$ a solution of $(\partial_{\dot{x}} L)' = \partial_x L$, and $x = x_0 + \varepsilon y$ a perturbed solution. Assuming all the solutions are uniformly bounded and defined over the same time interval, show that

$$(\partial_{\dot{y}}\tilde{L})' = \partial_y \tilde{L} + o(1) \text{ as } \varepsilon \to 0$$

where $\tilde{L} = \tilde{L}(t, y, \dot{y})$ is the time dependent quadratic function

$$\tilde{L} = \frac{1}{2}\partial_x^2 L(x_0(t), \dot{x}_0(t))|y|^2 + \partial_x\partial_{\dot{x}} L(x_0(t), \dot{x}_0(t))y \cdot \dot{y} + \frac{1}{2}\partial_{\dot{x}}^2 L(x_0(t), \dot{x}_0(t))|\dot{y}|^2$$

The approximate equation for y is the linearization of the Euler-Lagrange equation around x_0.

252. (Huygens Systems) Let $f : \mathbb{R}^d \to \mathbb{R}$ smooth. Given a smooth curve $\gamma : [0,1] \to \mathbb{R}^d$, let $\tilde{\gamma} = (\gamma, f \circ \gamma)$ be a lifted curve on the graph of f and

$$E(\gamma) = \frac{1}{2}\int_0^1 |\tilde{\gamma}'|^2 = \frac{1}{2}\int_0^1 |\gamma'|^2 + (Df(\gamma)\cdot\gamma')^2 \text{ (kinetic energy)},$$

$$L(\gamma) = \int_0^1 |\tilde{\gamma}'| = \int_0^1 \sqrt{|\gamma'|^2 + (Df(\gamma)\cdot\gamma')^2} \text{ (length)}$$

(1) Compute the Euler-Lagrange equations for E and L.

(2) Given that the functionals are time independent, they admit a law of conservation of energy. Compute these laws in each case.

(3) Show that if γ minimizes E, then $|\gamma'|$ is constant.

(4) Show that L is invariant by reparametrizations. For any increasing diffeomorphism $s : [0,1] \to [0,1]$ it holds that $L(\gamma) = L(\gamma \circ s)$.

(5) Show that $2E \geq L^2$ and equality holds if and only if $|\gamma'|$ is constant.

(6) Show that if γ minimizes E, then also minimizes L.

5.2. **Holonomic constrains 248.** Consider $q = (x,y) \in \mathbb{R}^{d-k} \times \mathbb{R}^k$ and $U : \mathbb{R}^{d-k} \times \mathbb{R}^k \to \mathbb{R}$ a smooth potential. The goal is to see that the dynamics of a mechanical system constrained to $\mathbb{R}^{d-k} = \{y = 0\}$ are given by

$$x'' = -\partial_x U(x, 0)$$

which seems the natural algebraic manipulation to do when we impose $y = 0$. For a precise definition of a constrained system, compatible with the physical intuition, we take the limit (which we will see exists) of a certain family of dynamics.

For $N \geq 0$, consider the potential $V_N(x,y) = U(x,y) + (N/2)|y|^2$ and the mechanical system

$$q_N'' = -DV_N$$

subject to a fixed initial condition

$$q_N(0) = 0, \qquad q_N'(0) = (v_0, 0) \in \mathbb{R}^{d-k}$$

Intuitively, for N large there are strong forces which constrain the system to remain close to $\mathbb{R}^{d-k}$. We do the analysis in several steps left as exercises:

(1) There is some $\delta, C > 0$ independent of N such that the solution is defined for $|t| < \delta$ and*

$$\sup_{\substack{|t|\leq\delta \\ N\geq 0}} [|x_N(t)| + |x'_N(t)| + N^{1/2}|y_N(t)|] < C$$

As a consequence we get that $y_N \to 0$ uniformly over the time interval $[-\delta, \delta]$. However we still do not know if x_N has a limit, but we have a candidate.

(2) Let $x = x(t) \in \mathbb{R}^{d-k}$ solves

$$x'' = -\partial_x U(x, 0), \qquad x(0) = 0, \qquad x'(0) = v_0$$

Then $z_N = x_N - x$ satisfies the linearized problem

$$z''_N + A_N(t)z_N = -B_N(t)y_N(t)$$

with

$$A_N(t) = \int_0^1 \partial^2_{xx} U((1-s)x_N(t) + sx(t), 0)ds,$$

$$B_N(t) = \int_0^1 \partial^2_{xy} U(x_N(t), sy_N(t))ds$$

(3) Given that the coefficients are uniformly bounded we get that for some $C > 0$ independent of N.

$$\sup_{\substack{|t|<\delta \\ N\geq 0}} N^{1/2}|z_N(t)| \leq C$$

In conclusion, as $N \to \infty$, q_N converges uniformly in $[-\delta, \delta]$ to $q_\infty = (x_\infty, 0)$.

This procedure can be generalized to a smooth manifold of the form $M = \{g = 0\}$ with $g = (g_1, \ldots, g_k) \in C^1(\mathbb{R}^d)$ and $dg_1, \ldots, dg_k$ linearly independent over M. If $\varphi : \mathbb{R}^{d-k} \to M$ defines a system of coordinates then the dynamics are given by the pullback of the Lagrangian. That is to say that for

$$L(q, \dot{q}) = \frac{m}{2}|D\varphi(q)\dot{q}|^2 - U(\varphi(q))$$

the dynamics are given by

$$\frac{d}{dt}\partial_{\dot{q}} L = \partial_q L$$

This is a natural way to introduce Lagrangians more general that $L = (m/2)|v|^2 - U(x)$.

253. Consider a graph surface $S = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y = f(x)\}$ for some smooth function f. A bead is constrained to stay on S and it is influenced only by the force of gravity $-mge_{d+1}$. Compute the Euler-Lagrange equations and the law of conservation of energy of this system.

254. Consider a graph surface $S = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y = f(x)\}$ for some smooth function f. A bead is constrained to stay on S and it is influenced by a potential $U = U(y)$. Given a flow $\Phi_s : \mathbb{R}^d \to \mathbb{R}^d$ such that for all $s \in \mathbb{R}$

$$f \circ \Phi_s = f, \qquad D\Phi_s \in SO(d)$$

*Hint: Given the regularity hypotheses of the ODE, the only way the solution might not exists in a given interval is if it escapes to infinity.

show that the mechanical system has a conserved quantity and computed it. in terms of the vector field that generates Φ.

255. Consider a curve $\Gamma = \{x \in \mathbb{R}^d : x = \gamma(q) \text{ for some } q \in \mathbb{R}\}$ where $\gamma : \mathbb{R} \to \mathbb{R}^n$ is smooth and one to one with $\gamma' \neq 0$. A bead is constrained to stay on Γ and it is influenced only by the force of gravity $-mge_d$. Compute the Euler-Lagrange equations and the law of conservation of energy of this system.

256. Consider a family of graph surfaces $S_t = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y = f(x, t)\}$ for some time dependent smooth function f. A bead is constrained to stay on S_t at time t and it is influenced only by the force of gravity $-mge_{d+1}$. Compute the Euler-Lagrange equations of this system.

257. Consider a family of curves $\Gamma_t = \{x \in \mathbb{R}^d : x = \gamma(q, t) \text{ for some } q \in \mathbb{R}\}$ where $\gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^d$ is smooth and one to one with $d\gamma/ds \neq 0$. A bead is constrained to stay on Γ_t at time t and it is influenced only by the force of gravity $-mge_d$. Compute the Euler-Lagrange equations of this system.

258. Consider a spherical pendulum given by a unit mass constrained to the unit sphere $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ and to a potential of the form $U = U(q_3)$. Take as coordinates $q = r(\cos\theta, \sin\theta) \in \mathbb{R}^2 = \{q \in \mathbb{R}^3 : q_3 = 0\}$ with the stereographic projection from the north pole

$$\varphi(q) = q + \frac{r^2 - 1}{r^2 + 1}(e_3 - q)$$

(1) Deduce the following Lagrangian

$$L(r, r', \theta') = 2\frac{r'^2 + (r\theta')^2}{(r^2 + 1)^2} - U\left(\frac{r^2 - 1}{r^2 + 1}\right)$$

(2) Deduce the following conservation law for the angular momentum

$$\left(\frac{2r}{r^2 + 1}\right)^2 \theta' = M$$

(3) Deduce the following conservation law for the energy

$$2\left(\frac{r'}{r^2 + 1}\right)^2 + U\left(\frac{r^2 - 1}{r^2 + 1}\right) + \frac{M^2}{2}\left(\frac{r^2 + 1}{2r}\right)^2 = E$$

(4) Deduce the corresponding equations for $\rho = \arctan r$ in terms of $\cos(2\rho)$.

259. Consider an chain in $\mathbb{R}^2$ attached to the origin and modeled as an n-pendulum with equal masses $\Delta m = m/n$ and lengths $\Delta l = l/n$ between them. To be precise the positions of the particles satisfy for $\Theta_j = (\cos\theta_j, \sin\theta_j)^T$,

$$q_1 = \Delta l \Theta_1, \qquad q_{j+1} = q_j + \Delta l \Theta_{j+1}$$

Consider only the force of gravity acting over this chain.

PICTURE

(1) Show that the the Euler-Lagrange system of equations is

$$\left(\sum_{j=i}^{n} q_j' \cdot \Theta_i^\perp\right)' = -(n + 1 - i)g\sin\theta_i - \sum_{j=i}^{n} \theta_j' q_j' \cdot \Theta_j, \qquad \Theta_i^\perp = (-\sin\theta_i, \cos\theta_i)$$

(2) Integrate the system by implementing the corresponding Euler method.

(3) Deduce the following linearization around the equilibrium $\theta_1 = \ldots = \theta_n = 0$

$$\sum_{j=1}^{n} \min\left(n+1-i, n+1-j\right) \vartheta_j'' \Delta l = -(n-i+1) g \vartheta_i$$

(4) Show that the previous equation is equivalent to a discrete wave equation of the form $\vartheta'' = \omega^2 \Delta((I-X)\vartheta)$ where

$$\omega^2 = \frac{g}{l}, \qquad X = \frac{1}{n}\operatorname{diag}(0,1,\ldots,(n-1)) \qquad \Delta = n^2 \begin{pmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}$$

(5) Compute the eigenvalues and eigenvectors of $\Delta(I-X)$.

260. Let $g, U : \mathbb{R}^d \to \mathbb{R}$ smooth and $M = \{g = 0\}$ such that

$$\inf_{q \notin M} \frac{g(q)}{d(q,M)} > 0, \qquad U \geq 0$$

For $N \geq 0$, consider the potential $V_N = U + (N/2)g^2$ and the the mechanical system

$$m q_N'' = -DV_N$$

subject to a fixed initial condition

$$q_N(0) = q_0 \in M, \qquad q_N'(0) = v_0$$

(1) Show that there is some $\delta > 0$ independent of N such that the solution is defined for $|t| < \delta$ and

$$\sup_{|t|<\delta} |q_N(t)| < 1$$

(2) Show that there exists some $C > 0$ independent of N such that

$$\sup_{|t|<\delta} d(q_N(t), M) \leq C N^{-1/2}$$

261. Consider the problem of finding $x = x(t)$ minimizing

$$\Phi(x,y) = \int_{-1}^{1} \left(\frac{1}{2}\dot{x}^2 - x\right) dt$$

among all curves under the constrain $x \leq 0$ and boundary conditions $x(\pm 1) = -\varepsilon < 0$.

(1) Show that for ε sufficiently small

$$x(t) = \min(-(1/2)(t+1)^2 + v_0(t+1) - \varepsilon, -(1/2)(t-1)^2 - v_1(t-1) - \varepsilon)$$

for some $v_0, v_1 \in \mathbb{R}$ such that $\max_{t \in (-1,1)} x(t) = 0$.

(2) Determine a constrained optimization problem in terms of (v_0, v_1) and deduce that $x(t)$ bounces at $\{x = 0\}$ with equal incoming and outgoing angles (i.e. $v_0 = v_1$).

(3) Generalize this result to constrains of the form $x(t) \leq f(t)$.

262. Let $L : \mathbb{R}^d \times \mathbb{R}^d \to R$ and for $N \geq 0$ consider the Lagrangian $L_N(q, \dot{q}) = L(q, \dot{q}) + (N/2)\max(0, q_d)^2$. Let $q_N = q_N(t)$ be the minimizer of

$$\Phi_N(q) = \int_0^1 L_N(q(t), \dot{q}(t))dt$$

with the boundary conditions

$$q_N(0) = q_0 \in \{q_d < 0\}, \qquad q_N(1) = q_1 \in \{q_d < 0\}$$

Assume that $q_N \to q_\infty$ uniformly in $[0, 1]$.

(1) Show that q_∞ is a minimizer of

$$\Phi(q) = \int_0^1 L(q, \dot{q}) \text{ subject to } q \in \{q_d < 0\}, q(0) = q_0, q(1) = q_1$$

(2) Show that if $q_\infty(t^*) \in \{q_d = 0\}$ then

$$\lim_{\varepsilon \to 0} q'_\infty(t^* - \varepsilon) + q'_\infty(t^* + \varepsilon) = 0$$

(3) Generalize these results for a constrain of the form $q_d(t) \leq f(q_1(t), \ldots, q_{d-1}(t))$.

263. (Brachistochrone) Consider $y = y(x) : [0, 1] \to \mathbb{R}$ such that $y(0) = 1$ and $y(1) = 0$. Let $T(y)$ be the time it takes a unit mass bead constrained to remain on the graph of y to go from $(0, 1)$ at rest to $(1, 0)$ by the action of gravity.

(1) Deduce that

$$T(y) = \int_0^1 \sqrt{\frac{1 + \dot{y}^2}{2gy}} dx$$

(2) Deduce that a minimizer of T satisfies

$$y(\dot{y}^2 + 1) = 1$$

(3) Integrate the differential equation to find the minimizing graph.

5.3. **Rigid bodies.** Consider a (Radon) measure ρ in $\mathbb{R}^d$. For any $E \subseteq \mathbb{R}^d$ (Borel) we interpret $\rho(E)$ as the amount of mass a rigid body has in E. This describes a stationary model.

To get all the possible configurations we take $M = SO(d) \times \mathbb{R}^d$ mapping a fixed reference frame $\mathbb{R}^d_Q$ to a moving reference frame $\mathbb{R}^d_q$ the following way: Given $D = (B, D) \in M$ and $Q \in \mathbb{R}^d$ then

$$Q \mapsto q = DQ = BQ + r$$

The measure ρ gets pushed forward to a measure $D_\# \rho$ defined by

$$D_\# \rho(E) = \rho(D^{-1}E) \qquad \Leftrightarrow \qquad \int \varphi d D_\# \rho = \int \varphi \circ D d\rho, \ \forall \varphi \in C_0(\mathbb{R}^d)$$

We assume that ρ has finite first moment, such that the following quantities are absolutely convergent

$$\mu = \int d\rho \text{ (total mass)}, \qquad CM = \frac{1}{\mu} \int Q d\rho \text{ (center of mass)}.$$

Moreover to define the kinetic energy we also need finite second moments as we will see in a moment.

Given $\dot{D} \in T_D M$ the action induces a vector field over $\mathbb{R}^d_q$ by

$$q \mapsto \dot{q} = \dot{D}Q = B\dot{\Omega}Q + \dot{r}, \qquad (Q = D^{-1}q)$$

where $\dot{\Omega} \in so(d)$ (skew-symmetric) such that $B\dot{\Omega} \in T_B SO(d)$. From now on we identify $so(d)$ with $\mathbb{R}^{\binom{d}{2}}$ given that any $\dot{\Omega} \in so(d)$ is uniquely determined by the $\binom{d}{2}$ numbers above its main diagonal.

We compute the kinetic energy from

$$T = \frac{1}{2}\int |\dot{q}|^2 dD_{\#}\rho = \frac{1}{2}\int |B\dot{\Omega}Q + \dot{r}|^2 d\rho = \frac{m}{2}|\dot{r}|^2 + m^2\dot{r} \cdot B\dot{\Omega}CM + \frac{1}{2}\int |\dot{\Omega}Q|^2 d\rho$$

In particular we could have arranged since the beginning $CM = 0$ and then obtain that T depends only on $\dot{r}$ and $\dot{\Omega}$

$$T = \frac{m}{2}|\dot{r}|^2 + \frac{1}{2}A(\dot{\Omega}), \qquad A(\dot{\Omega}) = A(\dot{\Omega}, \dot{\Omega}), \qquad A(\Omega_1, \Omega_2) = \int \Omega_1 Q \cdot \Omega_2 Q d\rho$$

The quadratic form A is known as the inertia operator.

Given $\Omega \in so(d)$ an $D = (B, r) \in M$ the angular momentum around Ω at $(\dot{B}, \dot{r}) = (B\dot{\Omega}, \dot{r}) \in T_B SO(d) \times T_r\mathbb{R}^d$ is defined by

$$I_\Omega = A(B\dot{\Omega}, \Omega B) = \int (\dot{q} - \dot{r}) \cdot \Omega(q - r) dD_{\#}\rho$$

In particular for $d = 3$ let us recall that

$$\Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \quad \Rightarrow \quad \Omega Q = \omega \times Q \text{ and } I_\Omega = \omega \cdot m$$

where

$$m = \int (q - r) \times (\dot{q} - \dot{r})\ dD_{\#}\rho \text{ angular momemtum around the center of mass}$$

264. Show that $\varphi : so(d) \times \mathbb{R}^d \to M = SO(d) \times \mathbb{R}^d$ given by $\varphi(\Omega, r) = (\exp(\Omega), r)$ defines a chart on M. In particular, the Lagragian corresponding the the free movement of the rigid body

$$L = T = \frac{m}{2}|\dot{r}|^2 + \frac{1}{2}\int |\dot{\Omega}Q| d\rho$$

has r as cyclic coordinate which implies the conservation of linear moment (i.e. the center of mass moves with constant velocity). Why is not Ω a cyclic coordinate?*

265. Given a measure ρ with finite first moment we define

$$m(\rho) = \int d\rho \text{ (total mass)}, \qquad CM(\rho) = \frac{1}{m}\int Q d\rho \text{ (center of mass)}$$

Given also $T : \mathbb{R}^d \to \mathbb{R}^{d'}$ Borel measurable we define the push-forward $T_{\#}\rho$ by

$$T_{\#}\rho(E) = \rho(T^{-1}E)\ \forall E \in \mathcal{B}(\mathbb{R}^{d'})$$

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*Hint: If $\varphi(\Omega) = \exp \Omega$ is the given chart, $D\varphi(\Omega)\dot{\Omega}$ still depends on Ω. There is not a simple expression for $D\varphi(\Omega)\dot{\Omega}$ unless Ω and $\dot{\Omega}$ commute.

(1) Show that

$$CM(\rho_1+\rho_2) = \frac{m(\rho_1)}{m(\rho_1)+m(\rho_2)}CM(\rho_1) + \frac{m(\rho_2)}{m(\rho_1)+m(\rho_2)}CM(\rho_2)$$

(2) Show that $m(T_\# \rho) = m(\rho)$.
(3) Given $T(Q) = Q - CM(\rho)$, show that $CM(T_\#\rho) = 0$. This allows one to assume $CM = 0$ up to a conveniently chosen translation.
(4) Given $T : \mathbb{R}^d \to \mathbb{R}^d$ convex, show that $CM(T_\#\rho) \leq T(CM(\rho))$.

266. Given $\Omega \in so(d)$ let $\Phi_s : SO(d) \to SO(d)$ defined by

$$\Phi_s(B) = \exp(\Omega s)B.$$

(1) Show that Φ is a flow and that it is generated by the vector field $V(B) = \Omega B \in T_B SO(d)$.
(2) Show that the kinetic energy admits the symmetry given by Φ, hence $I_\Omega(\dot\Omega) = A(\dot\Omega, B\Omega)$, the angular momentum around Ω is conserved. Moreover, for $d = 3$, the angular momentum m is conserved.
(3) Let $M = D^{-1}m = B^{-1}(m - r)$ be the angular momentum is the frame of reference that travels with the body. Establish the following equations for the movement of a free rigid body (known as Euler's equations)

$$\dot M + \dot\Omega M = 0$$

In particular, for $d = 3$

$$\dot M = M \times \dot\omega$$

6. A crash course on advanced calculus

Recommended lectures:

- Chapter 5 and 6 from Spivak's book [13].
- Section 38 from Arnold's mechanics book [2].

6.1. **Differential forms.** A differential form on $U \subseteq \mathbb{R}^d$ of order k is an integrand which purpose is to be integrated over a parametric surface of dimension k. Algebraically it generalizes the determinant in the sense that it is multilinear and alternating*.

We denote by $\Lambda^k(\mathbb{R}^d)$ the real vector space of multilinear functions from k copies of $\mathbb{R}^d$ to $\mathbb{R}$ which are alternating. The second construction is given by (smooth) fields of $\Lambda^k(\mathbb{R}^d)$ over U, in other words we consider $\Omega^k(U) = \{\omega : U \to \Lambda^k(\mathbb{R}^d)\}$ which we call k-differential forms.

The canonical basis of $\Lambda^k(\mathbb{R}^d)$ is given by $\{e_{j_1,\ldots,j_k} \mid 1 \leq j_1 < \ldots < j_k \leq d\}$ defined by

$$e_{j_1,\ldots,j_k}(v_1,\ldots,v_k) = \det(e_{j_\alpha} \cdot v_\beta)$$

This implies that $\dim \Lambda^k(\mathbb{R}^d) = \binom{d}{k}$ (or just 0 if $k > d$). By convention $\Lambda^0(\mathbb{R}^d) = \mathbb{R}$ and $\Omega^0(U) = C^\infty(U)$.

In the particular case when $k = d$, $\Lambda^d(\mathbb{R}^d)$ is generated by the determinant (the only d-multilinear alternating function in $\mathbb{R}^d$ for which $\omega(e_1,\ldots,e_d) = 1$). This means that every

*Alternating means that for any permutation $\sigma \in S_k$

$$\omega_x(v_{\sigma_1},\ldots,v_{\sigma_k}) = \operatorname{sign}\sigma\,\omega_x(v_1,\ldots,v_k)$$

$\omega \in \Omega^d(U)$ can also be associated with a function from U to $\mathbb{R}$ such that (with abuse of notation)

$$\omega_x = \omega(x) \det$$

In this case of maximal dimension, it is almost tautological how to define the integral of the form $\omega \in \Omega^d(U)$ with $U \subseteq \mathbb{R}^d$

$$\int_U \omega = \int_U \omega(x) dx$$

Given $f : V \to U$ we define the pull back $f^* : \Omega^k(U) \to \Omega^k(V)$ by

$$(f^*\omega)_x(v_1, \ldots, v_k) = \omega_{f(x)}(Df(x)v_1, \ldots, Df(x)v_k)$$

This is a contravariant construction (it satisfies an identity similar to the transpose of a matrix)

$$(f \circ g)^* = g^* \circ f^*$$

Given the parametric surface $c : V \subseteq \mathbb{R}^k \to U \subseteq \mathbb{R}^d$ we define the integral of $\omega \in \Omega^k(U)$ over c by the formula of change of variables

$$\int_c \omega = \int_V c^* \omega$$

This construction already implies the actual formula of change of variables

$$\int_{f \circ c} \omega = \int_c f^* \omega$$

267. Let $x \in U \subseteq \mathbb{R}^d$, $v_1, \ldots, v_k \in \mathbb{R}^d$ and $\varepsilon > 0$ sufficiently small such that the following $c_\varepsilon : [0,1]^k \to U$ is well defined

$$c_\varepsilon(t_1, \ldots, t_k) = x + \varepsilon(t_1 v_1 + \ldots + t_k v_k)$$

Given $\omega \in \Omega^k(U)$ show that

$$\lim_{\varepsilon \to 0} \varepsilon^{-k} \int_{c_\varepsilon} \omega = \omega_x(v_1, \ldots, v_k)$$

268. For $\omega_1, \omega_2 \in \Omega^k(U)$ show that $\omega_1 = \omega_2$ if and only if

$$\int_c \omega_1 = \int_c \omega_2 \text{ for all } c : [0,1]^k \to U$$

One of the main achievements of this construction is to identify a connection with some type of derivative, or just a Fundamental Theorem of Calculus. Before proceeding to its formulation we need a a couple of tools.

The wedge product: A natural way to define a product of forms $\omega \in \Lambda^k(\mathbb{R}^d)$ and $\eta \in \Lambda^l(\mathbb{R}^d)$ is just to take

$$(\omega \otimes \eta)(v_1, \ldots, v_{k+l}) = \omega(v_1, \ldots, v_k)\eta(v_{k+1}, \ldots, v_{k+l})$$

the problem is that this operation does not give an alternating form. To fix this we consider instead

$$(\omega \wedge \eta)(v_1, \ldots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sign} \sigma \omega(v_{\sigma_1}, \ldots, v_{\sigma_k})\eta(v_{\sigma_{k+1}}, \ldots, v_{\sigma_{k+l}})$$

The factor $1/(k!l!)$ can be roughly explained taking into consideration that given the permutation $\sigma \in S_{k+l}$ there are $k!$ way to reorder $(\sigma_1, \ldots, \sigma_k)$ and $l!$ ways to reorder $(\sigma_{k+1}, \ldots, \sigma_{k+l})$.

Some properties:

(1) The product is distributive.
(2) $\omega \in \Lambda^k(\mathbb{R}^d)$ and $\eta \in \Lambda^l(\mathbb{R}^d)$: $\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega$.
(3) The product is associative.

The fun one to prove is the associativity (see Chapter 5 from [13]).

6.2. **The Lie Derivative.**

7. Hamiltonian mechanics

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